8th World Congresses on Structural and Multidisciplinary Optimization
June 1-5, 2009, Lisbon, Portugal

Comparison Study between Probabilistic and Possibilistic Approach for Problems with Correlated Input and Lack of Input Statistical Information

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1. Abstract

Due to expensive experimental testing costs, in most industrial engineering applications, only limited statistical information is available to describe the input uncertainty model. It would be unreliable to use an estimated input uncertainty model, such as distribution types and parameters including the standard deviations for the distributions, that is obtained from insufficient data for the design optimization. Furthermore, when input variables are correlated, we would obtain non-optimum design if we use the assumption of independency for the design optimization.

In this paper, two methods for problems with lack of input statistical information – reliability-based design optimization (RBDO) with confidence level on the input model and possibility-based design optimization (PBDO) – are compared using a mathematical example and Abrams roadarm of an M1A1 tank. The comparison study shows that the PBDO could provide an unreliable optimum design when the number of samples is very small and that it provides an optimum design that is too conservative when the number of samples is relatively large. Furthermore, the optimum design does not converge to the optimum design obtained using the true input distribution as the number of samples increases. On the other hand, the RBDO with confidence level on the input model provides a reliable optimum design in a stable and consistent manner, and the optimum design converges to the optimum design obtained using the true input distribution as the number of samples increases.

2. Keywords: Identification of Joint and Marginal CDFs, Reliability-Based Design Optimization (RBDO), Possibility-Based Design Optimization (PBDO), Correlated Input, Copula

3. Introduction

Extensive research on reliability analysis and design optimization has been performed under the premise that an accurate input statistical model, such as the probability density function (PDF) or the cumulative distribution function (CDF), is available. Monte Carlo simulation (MCS) [1] is a very common and powerful method for estimating the reliability of a system. However, MCS requires a very large number of samples to accurately estimate the reliability, and this is not acceptable in large-scale industrial engineering applications. On the other hand, when applicable, MCS has been used as a benchmark test to compare the accuracy of a method. The importance sampling method [2] is also a simulation-based method; it requires a smaller number of samples than MCS but still a very large number of samples. As analytical methods for reliability analysis and reliability-based design optimization (RBDO), the first-order reliability method (FORM) [1,3-5] and the second-order reliability method (SORM) [6,7] have been widely used. To overcome the weaknesses of FORM and SORM, the most probable point (MPP)-based dimension reduction method (DRM) has recently been proposed [8,9]. As mentioned earlier, all these methods can be used for reliability estimation and design optimization only when the accurate input statistical model is available.

Theoretically, infinite raw data are necessary to exactly estimate the input statistical model. However, in many industrial engineering applications, it is difficult or impossible to have sufficient raw data for the accurate estimation of the input statistical model. In a case where it is not possible to estimate the accurate input statistical model, the probability-based methods may not be appropriate since improper modeling of input uncertainty could affect the system output more significantly than physical uncertainty does [10,11]. In such a case, possibility-based analysis and design optimization (PBDO) [12-15] or RBDO with confidence level on the input model [1,16,17] can be used to ensure a conservative optimum design.

Hence, the main focus of this paper is to compare two methods in terms of how conservative and stable the methods are for problems with lack of input statistical information, especially when the inputs are correlated. To apply both methods to the design optimization, it is necessary to estimate the input statistical model using a limited number of data. For this purpose, the Bayesian method [18,19] is introduced to identify the input marginal and joint CDF. After the estimation of the input statistical model, the RBDO with confidence level on the input model uses the upper bound of confidence interval of the estimated standard deviation, whereas the PBDO uses the estimated standard deviation but a different transformation. Since the existing possibility-based studies mentioned above are based on the assumption that input uncertainties are non-interactive, which corresponds to "independent" in the reliability theory, it is necessary to propose a new transformation for the interactive fuzzy variables.

In this paper, a mathematical example is used to show how two methods work in a problem with a lack of information and correlated input variables. The results of the two methods are compared with the MCS result. In addition, since certain fatigue material properties are known to be negatively correlated [20-22], a real engineering example, the Abrams roadarm of an M1A1 tank, which has fatigue life constraints, is also used to compare two methods. In this case, due to the expense of the simulation, MCS cannot be carried out for the benchmark result. Instead, a simple cost comparison is used to judge which method is more conservative. Section 4 briefly explains how to estimate the input statistical model using the Bayesian method. Sections 5 and 6 illustrate two methods, RBDO with confidence level on the input model and PBDO. Finally, two methods are compared using a mathematical example and an engineering problem in Section 7, followed by conclusions in Section 8.

4. Estimation of Input Statistical Model

Before the RBDO or PBDO is carried out, it is necessary to accurately identify the input statistical model, such as marginal CDFs, the joint CDF, and their parameters, using the limited experimental data. For the effective identification of the input marginal and joint CDFs, the Bayesian method is introduced and explained in Section 4.1. Section 4.2 illustrates how to quantify distribution parameters such as mean, standard deviation, and correlation coefficient.

4.1. Identification of Marginal and Joint CDFs Using the Bayesian Method

Consider a finite set $s_q \subset s$ consisting of candidates M_k, $k = 1,\cdots,q,$ where *s* is a set of all candidates and *q* is the number of the candidates. The Bayesian method consists of defining *q* hypotheses:

h_k : The data come from candidates M_k, $k = 1,\cdots,q$

The probability of each hypothesis h_k given the data *D* is defined as [18,19]

$$\Pr\left(h_k \middle| D, I\right) = \frac{\Pr\left(D \middle| h_k, I\right)\Pr\left(h_k \middle| I\right)}{\Pr\left(D \middle| I\right)} \tag{1}$$

where $\Pr\left(D \middle| h_k, I\right)$ is the likelihood function, $\Pr\left(h_k \middle| I\right)$ is the prior on the candidate, and $\Pr\left(D \middle| I\right)$ is the normalization constant with any relevant additional knowledge *I*. Equation (1) can be rewritten in an integration form as

$$\begin{aligned} \Pr\left(h_k \middle| D, I\right) &= \int_{-\infty}^{\infty} \Pr\left(h_k, \gamma \middle| D, I\right) d\gamma \\ &= \int_{-\infty}^{\infty} \frac{\Pr\left(D \middle| h_k, \gamma, I\right)\Pr\left(h_k \middle| \gamma, I\right)\Pr\left(\gamma \middle| I\right)}{\Pr\left(D \middle| I\right)} d\gamma \end{aligned} \tag{2}$$

where γ is a parameter such as mean and standard deviation for the identification of marginal distributions and a correlation coefficient for the identification of a joint distribution.

For the input marginal distribution, seven candidates, which are Gaussian, Weibull, Gamma, Lognormal, Gumbel, Extreme, and Extreme type II distributions, are used. For the joint distribution, eight candidate copulas (*C*), which are Clayton, AMH, Gumbel, Frank, A12, A14, FGM, and Gaussian, are introduced to model the joint distribution [23-25].

Under the hypothesis h_k that the data *D* come from the candidate M_k, the likelihood function in Eq. (2), which is the probability of drawing the data *D* for the hypothesis on M_k, is expressed as

$$\Pr\left(D \middle| h_k, \gamma, I\right) = \prod_{i=1}^{ns} f_k\left(x_i \middle| \gamma\right) \tag{3}$$

for a marginal PDF f_k and

$$\Pr\left(D \middle| h_k, \gamma, I\right) = \prod_{i=1}^{ns} c_k\left(u_i, v_i \middle| \gamma\right) \tag{4}$$

for a copula density function c_k, which is defined as

$$c_k\left(u, v\right) = \frac{\partial^2 C_k\left(u, v\right)}{\partial u \partial v} \tag{5}$$

for bivariate data. In Eq. (3), x_i is the i$^{\text{th}}$ sample value and *ns* is the number of samples; in Eq. (4), u_i and v_i are the marginal CDF values defined as $u_i = F_X\left(x_i\right)$ and $v_i = F_Y\left(y_i\right)$ and obtained from the given paired data $\left(x_i, y_i\right)$. Since the paired data are independent of each other in Eqs. (3) and (4), the likelihood function is expressed as multiplications of the marginal PDFs and copula density function values evaluated at all the data.

Let the additional information I be as follows [18,19]:

I_1: A parameter γ belongs to the set Λ^{γ}, and each estimated parameter is equally likely.

I_2: For a given parameter, all candidates satisfying $\gamma \in \Omega_k^{\gamma}$ are equally probable where Ω_k^{γ} are domains of γ for M_k.

The set Λ^{γ} provides information on the interval of γ that a user might know. For example, if the user knows the specific domain of Λ^{γ}, the domain can be used to integrate the likelihood function. However, if any information on γ is not provided, it might be assumed as $\Lambda^{\gamma} = (-\infty, \infty)$ for mean, $\Lambda^{\gamma} = (0, \infty)$ for standard deviation, and $\Lambda^{\gamma} = [-1, 1]$ for Kendall's tau. For the mean and standard deviation, the infinite domain cannot practically be used to integrate the likelihood function, and thus the finite range of Λ^{γ} needs to be determined from samples such that Λ^{γ} covers a wide range of γ.

Using the first additional information I_1, the prior on γ can be defined as

$$\Pr(\gamma | I_1) = \begin{cases} \dfrac{1}{\lambda(\Lambda^{\gamma})} & \gamma \in \Lambda^{\gamma} \\ 0 & \gamma \notin \Lambda^{\gamma} \end{cases} \tag{6}$$

where $\lambda(\Lambda^{\gamma})$ is the Lebesgue measure and is defined as the interval width of Λ^{γ}. Likewise, since all candidates M_k are equally probable for $\gamma \in \Omega_k^{\gamma}$, the prior on M_k is defined as

$$\Pr(h_k | \gamma, I_2) = \begin{cases} 1 & \gamma \in \Omega_k^{\gamma} \\ 0 & \gamma \notin \Omega_k^{\gamma} \end{cases} \tag{7}$$

In this paper, it is assumed that the prior follows a uniform distribution, which means there is no information on the distribution of γ.

The normalization constant $\Pr(D|I)$ can be expressed as

$$\Pr(D|I) = \sum_{k=1}^{q} \Pr(D | h_k, \gamma, I) \Pr(h_k | \gamma, I) \tag{8}$$

Since the normalization constant does not affect the identification of the input statistical model, it is not used in Eq. (2). Substituting all the equations, Eq. (2) can be rewritten as

$$\Pr(h_k | D, I) = \frac{\int_{\Omega_k^{\gamma} \cap \Lambda^{\gamma}} \prod_{i=1}^{ns} f_k(x_i | \gamma) d\gamma}{\lambda(\Lambda^{\gamma})} \tag{9}$$

for the marginal CDF identification and

$$\Pr(h_k | D, I) = \frac{\int_{\Omega_k^{\gamma} \cap \Lambda^{\gamma}} \prod_{i=1}^{ns} c_k(u_i, v_i | \gamma) d\gamma}{\lambda(\Lambda^{\gamma})} \tag{10}$$

for the joint CDF identification. The value calculated from Eqs. (9) and (10) is called the weight (W_k) and is used to identify the marginal and joint CDFs. The largest weight means that the candidate marginal or joint CDF is best fitting the given data set, and the candidate with the largest weight is selected to describe the input statistical model. For ease of understanding, the normalized weight of each candidate is used in this paper and is defined as

$$w_k = \frac{W_k}{\sum_{i=1}^{q} W_k}. \tag{11}$$

4.2. Quantification of Distribution Parameters

Once the marginal and joint CDFs are identified using the Bayesian method, it is necessary to quantify their parameters based on the given data. The mean and variance that are estimated from the given data are called the sample mean $\bar{x}$ and variance s^2 and are given by

$$\bar{x} = \sum_{i=1}^{ns} \frac{x_i}{ns} \tag{12}$$

and

$$s^2 = \frac{1}{ns-1}\sum_{i=1}^{ns}\left(x_i - \bar{x}\right)^2 \tag{13}$$

respectively. To calculate the correlation parameter θ from the given data, first, the sample version of Kendall's tau is calculated as [23]

$$\tau = \frac{c-d}{c+d} = \frac{2(c-d)}{ns(ns-1)} \tag{14}$$

where c is the number of concordant pairs and d is the number of discordant pairs. Then, θ is calculated using the explicit formulation for each copula [23-25].

For the calculation of the confidence interval of standard deviation, suppose that X is a Gaussian random variable and the samples come from *ns* independent Gaussian random variables, i.e., X_1, $X_2, \cdots$, X_{ns}, and let μ and σ^2 be a population mean and variance, respectively. From Eq. (13), the sample variance, which is also a random variable, can be rewritten as

$$\left(ns-1\right)S^2 = \sum_{i=1}^{ns}\left(X_i - \mu\right)^2 - ns\left(\bar{X} - \mu\right)^2 \tag{15}$$

Dividing both sides of Eq. (15) by σ^2, Eq. (15) is written as

$$\frac{\left(ns-1\right)S^2}{\sigma^2} = \sum_{i=1}^{ns}\left(\frac{X_i - \mu}{\sigma}\right)^2 - \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{ns}}\right)^2 \tag{16}$$

Since the first term and the second term of the right side of Eq. (16) follow a chi-square distribution with *ns* and one degree of freedom, respectively. The left side of Eq. (16) also follows a chi-square distribution with *ns*−1 degree of freedom, denoted as χ^2_{ns-1} [1].

When the PDF has a chi-square distribution with *ns*−1 degrees of freedom, the two-sided (1−α)% confidence interval for σ^2 is given as

$$\Pr\left[c_{\alpha/2,ns-1} \le \frac{\left(ns-1\right)S^2}{\sigma^2} \le c_{1-\alpha/2,ns-1}\right] = 1-\alpha \tag{17}$$

where c can be obtained from the chi-square distribution table using the given α and degrees of freedom. Using the realization of S^2, denoted as s^2, Eq. (17) is written as

$$\Pr\left[\frac{\left(ns-1\right)s^2}{c_{1-\alpha/2,ns-1}} \le \sigma^2 \le \frac{\left(ns-1\right)s^2}{c_{\alpha/2,ns-1}}\right] = 1-\alpha \tag{18}$$

Thus, the lower and upper bounds of the (1−α)% confidence interval for the standard deviation are calculated as

$$\sigma^{\mathrm{L}}_{1-\alpha} = \sqrt{\frac{\left(ns-1\right)s^2}{c_{1-\alpha/2,ns-1}}} \tag{19}$$

and

$$\sigma^{\mathrm{U}}_{1-\alpha} = \sqrt{\frac{\left(ns-1\right)s^2}{c_{\alpha/2,ns-1}}}, \tag{20}$$

respectively. The upper bound in Eq. (20) is used for the RBDO with confidence level on the input model, which will be explained in Section 5, to have the required confidence level on the input model and thus obtain a conservative optimum design for problems with lack of input statistical information.

5. Probabilistic Approach: RBDO with Confidence Level

The RBDO problem can be formulated to

$$\begin{aligned} &\text{minimize} \quad \text{Cost}(\mathbf{d}) \\ &\text{subject to} \quad P(G_i(\mathbf{X}) > 0) \le P_{F_i}^{Tar},\ i = 1,\cdots,nc \\ &\mathbf{d}^{\mathbf{L}} \le \mathbf{d} \le \mathbf{d}^{\mathbf{U}}, \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr} \end{aligned} \tag{21}$$

where $\mathbf{X}$ is the vector of random variables; $\mathbf{d}$ is the vector of design variables, which is the mean value of $\mathbf{X}$, $\mathbf{d} = \mu(\mathbf{X})$; $P(\bullet)$ is a probability measure; $G_i(\mathbf{X})$ represents the i^th^ constraint functions and constraint is defined as failure if $G_i(\mathbf{X}) > 0$; $P_{F_i}^{Tar}$ is the given target probability of failure for the i^th^ constraint; and *nd, nr,* and *nc* are the number of design variables, random variables, and constraints, respectively.

Using the enhanced performance measure approach (PMA+) [26], the i^th^ constraint in Eq. (21) can be rewritten in a deterministic way as

$$P(G_i(\mathbf{X})>0)-P_{F_i}^{\mathrm{Tar}} \le 0 \Rightarrow G_i(\mathbf{x}_{\mathrm{R}}^*) \le 0 \tag{22}$$

where $G_i(\mathbf{x}_{\mathrm{R}}^*)$ is the i$^{\text{th}}$ constraint function evaluated at the MPP, $\mathbf{x}_{\mathrm{R}}^*$, obtained from the inverse reliability analysis. To find the MPP, the Rosenblatt transformation [27] from the original space (X-space) into the standard Gaussian space (U-space) is required. Assuming that all input random variables are independent—that is, the joint CDF is given by the multiplication of the marginal CDFs—the transformation is given as

$$\Phi(u_i)=F_{X_i}(x_i) \text{ or } u_i=\Phi^{-1}\left[F_{X_i}(x_i)\right] \tag{23}$$

where $\Phi(u)=\int_{-\infty}^{u}\phi(\xi)d\xi$ and $\phi(u)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}u^2\right)$ are the CDF and PDF of the standard Gaussian random variable, respectively. For problems with lack of input statistical information, the upper bound of the standard deviation obtained using Eq. (20) will be used for the transformation, instead of using the estimated standard deviation. For example, if the Gaussian distribution is identified for a random variable X_i, then the transformation in Eq. (23) will be

$$u_i=\frac{x_i-\mu^e}{\sigma_{1-\alpha}^{\mathrm{U}}} \tag{24}$$

where μ^e is the estimated mean from the samples.

The Rosenblatt transformation for the correlated variables is obtained using the copula and marginal CDF estimated from the insufficient data as

$$\Phi(u_i)=F_{X_i}\left(x_i\middle|x_1,,\cdots,x_{i-1}\right)=C\left(F_{X_1}(x_1),\cdots,F_{X_i}(x_i)\middle|\boldsymbol{\theta}\right) \tag{25}$$

where $\Phi(\bullet)$ is the CDF of the standard Gaussian random variable, $F_{X_i}\left(x_i\middle|x_1,x_2,\cdots,x_{i-1}\right)$ is the conditional CDF of X_i obtained from the joint CDF by using $X_1=x_1$, $X_2=x_2$ $,\cdots,X_{i-1}=x_{i-1}$, C is the copula, and $\boldsymbol{\theta}$ is the matrix of the correlation parameters of $x_1,\cdots,x_n$. If only two variables, x_i and x_j, are correlated, then the transformation in Eq. (25) can be simplified as

$$\begin{cases}\Phi(u_i)=F_{X_i}(x_i)\\ \Phi(u_j)=F_{X_j}\left(x_j\middle|x_i\right)=C\left(F_{X_i}(x_i),F_{X_j}(x_j)\middle|\theta_{ij}\right)\end{cases} \tag{26}$$

or

$$\begin{cases}\Phi(u_i)=F_{X_j}(x_j)\\ \Phi(u_j)=F_{X_i}\left(x_i\middle|x_j\right)=C\left(F_{X_i}(x_i),F_{X_j}(x_j)\middle|\theta_{ij}\right)\end{cases} \tag{27}$$

based on different transformation orders. The effect of the transformation order is beyond the scope of this paper and explained in detail by Noh et al. [28]. For the correlated variables, the upper bound of the standard deviation in Eq. (20) is used for the transformation if a problem has lack of input statistical information.

Using the transformation explained above, the MPP in Eq. (22) can be obtained by solving the following optimization problem (inverse reliability analysis):

$$\begin{aligned}&\text{maximize} && g_i(\mathbf{u})\\ &\text{subject to} && \|\mathbf{u}\|_2=\beta_{t_i}\end{aligned} \tag{28}$$

where $g_i(\mathbf{u})$ is the i$^{\text{th}}$ constraint function in U-space defined as $g_i(\mathbf{u})\equiv G_i(\mathbf{x}(\mathbf{u}))=G_i(\mathbf{x})$ and β_{t_i} is the i$^{\text{th}}$ target reliability index such that $\beta_{t_i}=-\Phi^{-1}\left(P_{F_i}^{\mathrm{Tar}}\right)$. Then, using the MPP $\mathbf{x}_{\mathrm{R}}^*$ obtained from Eq. (28) and FORM, Eq. (21) can be rewritten using PMA+ in a deterministic manner as

$$\begin{aligned}&\text{minimize} && \mathrm{Cost}(\mathbf{d})\\ &\text{subject to} && G_i(\mathbf{x}_{\mathrm{R}}^*)\le 0,\ i=1,\cdots,nc\\ &&& \mathbf{d}^{\mathrm{L}}\le\mathbf{d}\le\mathbf{d}^{\mathrm{U}},\ \ \mathbf{d}\in R^{nd} \text{ and } \mathbf{X}\in R^{nr}\end{aligned} \tag{29}$$

However, since the FORM uses the linear approximation at the MPP, the probability of failure estimation using the FORM could very well be erroneous if the performance function is highly nonlinear or multi-dimensional or both. In such a case, a more accurate probability of failure estimation can be obtained using the MPP-based DRM and a more accurate MPP denoted by $\mathbf{x}_{\mathrm{DRM}}^*$ can be obtained using the accurate probability of failure [9]. Hence, for a system with highly nonlinear or multi-dimensional performance functions, Eq. (21) can be reformulated as

$$\begin{aligned} &\text{minimize} \quad \text{Cost}(\mathbf{d}) \\ &\text{subject to} \quad G_i(\mathbf{x}^*_{\text{DRM}}) \le 0, \; i = 1, \cdots, nc \\ &\qquad \mathbf{d}^{\text{L}} \le \mathbf{d} \le \mathbf{d}^{\text{U}}, \;\; \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr} \end{aligned} \tag{30}$$

6. Possibilistic Approach

When sufficient data are not available for modeling input CDFs, the probability-based method using the estimated standard deviation may not be appropriate since improper modeling of input uncertainty could affect the system output more significantly than the physical uncertainty does. In such a case, the possibility-based method where input variables are treated as fuzzy variables could be one option since the possibility-based method yields a more conservative optimum design than the probability-based method if the same input statistical model is used [11]. The difference between the RBDO with confidence level on the input model and the PBDO for a problem with a lack of input statistical information is that the RBDO with confidence level on the input model uses the upper bound of the standard deviation as an input, whereas the PBDO directly uses the estimated standard deviation but with a different transformation, which is explained in Section 6.2.

6.1. Formulation

For a system with a lack of information on input data, the PBDO problem can be formulated to

$$\begin{aligned} &\text{minimize} \quad \text{Cost}(\mathbf{d}) \\ &\text{subject to} \quad \Pi(G_i(\mathbf{X}) > 0) \le \alpha_{t_i}, \; i = 1, \cdots, nc \\ &\qquad \mathbf{d}^{\mathbf{L}} \le \mathbf{d} \le \mathbf{d}^{\mathbf{U}}, \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nf} \end{aligned} \tag{31}$$

where $\mathbf{X}$ is the vector of fuzzy variables where each fuzzy variable X_i has the membership function $\Pi_{X_i}(x_i)$ and the maximal grade $\max\{\Pi_{X_i}(x_i)\} = d_i$ [29]; $\Pi(\bullet)$ is a possibility measure; α_{t_i} is the target possibility of failure for the i$^{\text{th}}$ constraint; and *nf* is the number of fuzzy variables. Theories on possibility and fuzzy set are explained in detail by Du and Choi [11].

Similar to Eq. (22), using PMA+, the constraints in Eq. (31) are converted in a deterministic manner to

$$\Pi(G_i(\mathbf{X}) > 0) - \alpha_{t_i} \le 0 \Rightarrow G_i(\mathbf{x}^*_{\Pi}) \le 0 \tag{32}$$

where $\mathbf{x}^*_{\Pi}$ is the MPP obtained from the inverse possibility analysis given by

$$\begin{aligned} &\text{maximize} \quad \tilde{G}_i(\mathbf{v}) \\ &\text{subject to} \quad \|\mathbf{v}\|_\infty = 1 - \alpha_{t_i} \end{aligned} \tag{33}$$

where $\tilde{G}_i(\mathbf{v})$ is the i$^{\text{th}}$ constraint function in the standard normalized fuzzy V-space defined as

$$\tilde{G}_i(\mathbf{v}) \equiv G_i(\mathbf{x}(\mathbf{v})) = G_i(\mathbf{x}) \tag{34}$$

The transformation from X-space to V-space will be explained in Section 6.2 in detail for both non-interactive and interactive fuzzy variables.

In many industrial engineering applications, random and fuzzy variables may exist simultaneously. For example, in structural fatigue analysis, geometry variables can be treated as random variables since it is easy to handle the geometry during the manufacturing process. On the other hand, fatigue material properties can be treated as fuzzy variables since it is very expensive to measure the variability of fatigue material properties and oftentimes there are not sufficient data available. In a case that has random and fuzzy variables simultaneously, if all variables are treated as random, then the design optimization could show an unreliable optimum design due to improper modeling of input uncertainty. But if all variables are treated as fuzzy, then the optimization could yield too conservative an optimum design [11]. Hence, in such a case, it is desirable to use the mixed variable design optimization (MVDO) [14], which is formulated using PMA+ to

$$\begin{aligned} &\text{minimize} \quad \text{Cost}(\mathbf{d}) \\ &\text{subject to} \quad G_i(\mathbf{x}^*) \le 0, \; i = 1, \cdots, nc \\ &\qquad \mathbf{d}^{\text{L}} \le \mathbf{d} \le \mathbf{d}^{\text{U}}, \mathbf{d} \in R^{nd} \text{ and } \mathbf{X} \in R^{nr+nf} \end{aligned} \tag{35}$$

The MPP in Eq. (35) can be obtained by solving the optimization given by

$$\begin{aligned} &\text{maximize} \quad \tilde{g}_i(\mathbf{u}, \mathbf{v}) \\ &\text{subject to} \quad \|\mathbf{u}\|_2 = \beta_{t_i} = -\Phi^{-1}(\alpha_{t_i}) \\ &\qquad \|\mathbf{v}\|_\infty = 1 - \alpha_{t_i} \end{aligned} \tag{36}$$

where $\tilde{g}_i(\mathbf{u}, \mathbf{v})$ is the i$^{\text{th}}$ constraint function defined as

$$\tilde{g}_i(\mathbf{u}, \mathbf{v}) \equiv G_i(\mathbf{x}(\mathbf{u}), \mathbf{x}(\mathbf{v})) = G_i(\mathbf{x}) \tag{37}$$

6.2. Transformation

As explained in the previous section, to carry out the inverse possibility analysis, the fuzzy variables X_i need to be transformed to the standard normalized fuzzy variables V_i using the membership function. Thus, for the transformation, it is necessary to generate the membership function from the temporary CDF, which is the estimated CDF from the insufficient samples.

For non-interactive fuzzy variables, the membership function is generated to satisfy the probability-possibility consistency principle and the least conservative principle [13] as

$$\begin{aligned}\Pi_{X_i}(x_i) &= 1 - \left|2F_{X_i}(x_i) - 1\right| \\ &= \begin{cases} 2F_{X_i}(x_i) & x_i \in \{x_i : F_{X_i}(x_i) \le 0.5\} \\ 2 - 2F_{X_i}(x_i) & x_i \in \{x_i : F_{X_i}(x_i) > 0.5\} \end{cases}\end{aligned} \tag{38}$$

where $F_{X_i}(x_i)$ is the temporary CDF of X_i estimated from insufficient data. Then, the transformation from X_i to V_i can be written as

$$V_i = \begin{cases} \Pi_{X_i,L}(x_i) - 1 & , x_i \le d_i \\ 1 - \Pi_{X_i,R}(x_i) & , x_i > d_i \end{cases} \tag{39}$$

where $\Pi_{X_i,L}(x_i)$ and $\Pi_{X_i,R}(x_i)$ are the left side and right side of the membership function of the input fuzzy variable X_i, respectively, and d_i is the maximal grade of the membership function. Inserting Eq. (38) into Eq. (39) yields

$$V_i = \left\{ \begin{array}{l} \Pi_{X_i,L}(x_i) - 1 = 2F_{X_i}(x_i) - 1 \quad , x_i \le d_i \\ 1 - \Pi_{X_i,R}(x_i) = 2F_{X_i}(x_i) - 1 \quad , x_i > d_i \end{array} \right\} = 2F_{X_i}(x_i) - 1 \tag{40}$$

Figure 1 shows the hyper-cube for the inverse possibility analysis obtained using Eqs. (33) and (40) with the corresponding target possibility of failure $\alpha = 0.02275$, where both X_1 and X_2 follow the Gaussian distribution with a mean of 5 and a standard deviation of 0.3. From the figure, it can be seen that the hyper-cube is always larger than the hyper-sphere for the inverse reliability analysis if the same input statistical model is used, which guarantees that the possibility-based method is always more conservative than the probability-based method.

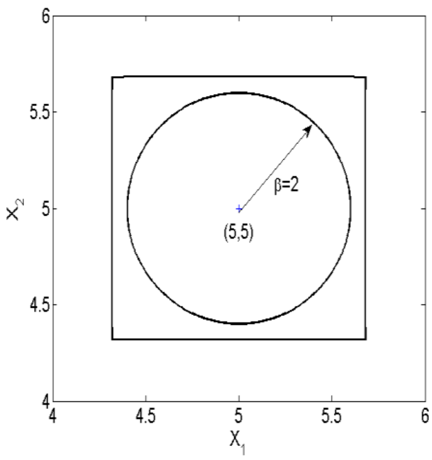

Figure 1. MPP Search Domain for Inverse Possibility and Probability Analysis

For interactive fuzzy variables, a new transformation from X-space to V-space is required for the inverse possibility analysis to find the MPP. Since the transformation for the non-interactive fuzzy variables can be readily used, as shown in Eq. (40), it is necessary to transform the interactive fuzzy variables **X** to the non-interactive fuzzy variables **Y** using Eq. (25). If only two variables, x_i and x_j, are interactive, then the transformation can be expressed as

$$\begin{cases} \Phi(y_i) = F_{X_i}(x_i) \\ \Phi(y_j) = F_{X_j}\left(x_j \middle| x_i\right) = C\left(F_{X_i}(x_i), F_{X_j}(x_j) \middle| \theta_{ij}\right) \end{cases} \tag{41}$$

or

$$\begin{cases} \Phi(y_i) = F_{X_j}\left(x_j\right) \\ \Phi(y_j) = F_{X_i}\left(x_i \middle| x_j\right) = C\left(F_{X_i}\left(x_i\right), F_{X_j}\left(x_j\right) \middle| \theta_{ij}\right) \end{cases} \tag{42}$$

Then, the non-interactive fuzzy variables **Y** are transformed to the non-interactive standard normalized fuzzy variables **V** using Eq. (40) as

$$\begin{cases} V_i = 2\Phi(y_i) - 1 = 2F_{X_i}\left(x_i\right) - 1 \\ V_j = 2\Phi(y_j) - 1 = 2C\left(F_{X_i}\left(x_i\right), F_{X_j}\left(x_j\right) \middle| \theta_{ij}\right) - 1 \end{cases} \tag{43}$$

or

$$\begin{cases} V_i = 2\Phi(y_i) - 1 = 2F_{X_j}\left(x_j\right) - 1 \\ V_j = 2\Phi(y_j) - 1 = 2C\left(F_{X_i}\left(x_i\right), F_{X_j}\left(x_j\right) \middle| \theta_{ij}\right) - 1 \end{cases} \tag{44}$$

for the bivariate data.

Figure 2 shows the MPP search domain for the interactive bivariate data where the true copula is the Clayton and the two marginal CDFs are normal. As shown in Fig. 2, the MPP search domain for the inverse possibility analysis is still larger than the domain for the inverse probability analysis even when the variables are correlated, which still guarantees that the possibilistic approach is always more conservative than the probabilistic approach if the same input statistical model is used. The two vertical lines along the X_2-axis in the MPP domain for the inverse possibility analysis in Fig. 2 appear because the transformation in Eq. (43) is used. If Eq. (44) is used, then the MPP search domain will have two horizontal lines along the X_1-axis. Hence, two different optimum results are obtained, depending on the transformation order used. The probabilistic approach can reduce the difference between two different transformation orders using the MPP-based DRM, whereas the difference cannot be reduced in the possibilistic approach.

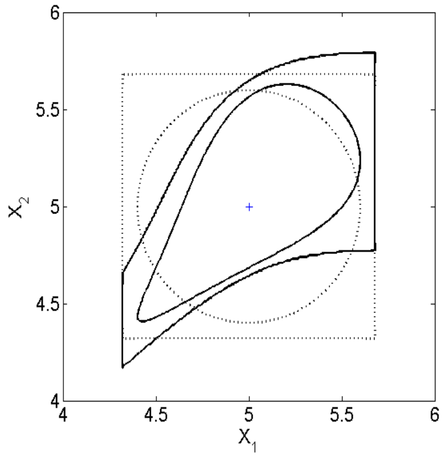

Figure 2. MPP Search Domain for Correlated Random and Fuzzy Variables

Since the transformation for the interactive fuzzy variables is now available, the design optimization can be formulated for a general case in which correlated or independent random variables and interactive or non-interactive fuzzy variables coexist. In such a case, the design optimization and MPP search can be still formulated using Eqs. (35) and (36), respectively. For the inverse analysis to find the MPP, which is carried out in the standard space (UV-space), the transformation in Eq. (23) for the independent random variables, the transformation in Eq. (26) or (27) for the correlated random variables, the transformation in Eq. (40) for the non-interactive fuzzy variables, and the transformation in Eq. (43) or (44) for the interactive fuzzy variables are used.

7. Numerical Examples

Numerical studies are carried out in this section to compare two approaches, PBDO and RBDO with confidence level on the input model, using a 2-D mathematical example and an M1A1 Abrams tank roadarm example. For both examples, a true input statistical model is assumed, and then a limited number of data is sampled from the true input statistical model. Using the sampled data, the input marginal and joint CDFs are identified. In addition, distribution parameters and their upper bounds are also quantified from the samples. PBDO and MVDO directly use the identified CDFs and estimated distribution parameters. On the other hand, RBDO with confidence level on the input model uses the identified CDFs and the upper bound of the estimated standard deviation for design optimization. In a 2-D mathematical example, results of both methods are compared with the MCS result to see which method yields an optimum design that is reliable but not too conservative. For this comparison, 100 data sets are used for each number of sampled data to see the statistical behavior of two methods. In an M1A1 Abrams tank roadarm example, since the MCS cannot be used due to its computational expense, the cost functions of both methods are compared with the cost function obtained using the true input statistical model.

7.1. 2-D Mathematical Example

Consider a 2-D mathematical example with a linear cost function and three constraints written as

$$\begin{aligned} & f(\mathbf{d}) = -d_1 + d_2 \\ & G_1(\mathbf{X}) = 1 - \frac{X_1^2 X_2}{20} \\ & G_2(\mathbf{X}) = 1 - \frac{(X_1 + X_2 - 5)^2}{30} - \frac{(X_1 - X_2 - 12)^2}{120} \\ & G_3(\mathbf{X}) = 1 - \frac{80}{X_1^2 + 8X_2 + 5} \\ & \mathbf{d}^{\text{initial}} = [5,5]^{\text{T}},\ \mathbf{d}^{\text{L}} = [0,0]^{\text{T}} \text{ and } \mathbf{d}^{\text{U}} = [10,10]^{\text{T}} \end{aligned} \tag{45}$$

The true input marginal distributions for Eq. (45) are assumed to be $X_1 \sim N(d_1, 0.3^2)$ and $X_2 \sim N(d_2, 0.3^2)$, respectively, and joint distribution is assumed to be the Clayton copula with the correlation coefficient $\tau = 0.5$. Both the marginal and joint CDFs are assumed to be unknown and must be identified using the Bayesian method explained in Section 4.1. However, in this example, the identification of the marginal and joint CDFs are assumed to be exact because the effect of the identification on two approaches are almost the same and the main focus of this example is to see the effect of the quantified distribution parameters on two different approaches with different numbers of samples. The target probability of failure is given as $P_{F_i}^{\text{Tar}} = 2.275\%$, and the target possibility of failure is also given as $\alpha_{t_i} = 0.02275$ for all constraints. Figure 3 shows the shape of the constraints in Eq. (45).

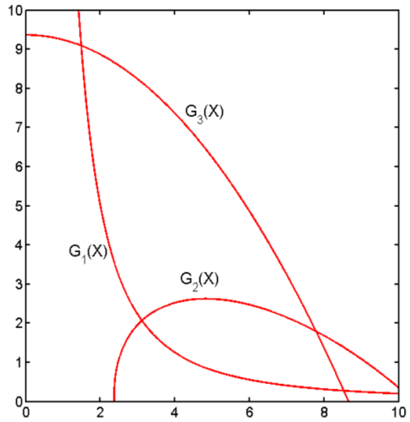

Figure 3. Constraint Shape of Eq. (45)

Table 1 shows the estimated distribution parameters—mean (μ_{est}) and standard deviation (σ_{est})—and the upper bounds of the standard deviation obtained using 95% confidence level on the input model (σ_{upp}) obtained from 100 sample sets. In Table 1, ns means the number of samples. From the table, it can be seen that the mean values of the estimated mean and standard deviation are very close to the true values, which are 5.000 and 0.3, respectively. However, the variation of the estimated values decreases as the number of samples increases. The same situation occurs in the estimation of the correlation coefficient as shown in Table 2, but the variation of the correlation coefficient is larger than mean and standard deviation.

Table 1. Estimated Parameters and Bounds (100 Sample Sets)

		X_1			X_2		
		Min.	Mean	Max.	Min.	Mean	Max.
μ_{est}	ns=10	4.8072	5.0073	5.2192	4.7894	4.9986	5.2305
	ns=50	4.8974	4.9978	5.0927	4.8902	4.9987	5.1055
	ns=100	4.9144	4.9950	5.0775	4.9238	4.9973	5.0639
	ns=500	4.9760	5.0000	5.0350	4.9681	5.0016	5.0316
σ_{est}	ns=10	0.1515	0.2955	0.5053	0.1599	0.2991	0.4513
	ns=50	0.2372	0.2970	0.3687	0.2251	0.2979	0.3935
	ns=100	0.2458	0.3000	0.3564	0.2513	0.3019	0.3662
	ns=500	0.2777	0.2993	0.3209	0.2771	0.3016	0.3257
σ_{upp}	ns=10	0.2766	0.5395	0.9224	0.2920	0.5460	0.8239
	ns=50	0.2956	0.3701	0.4594	0.2805	0.3712	0.4903
	ns=100	0.2856	0.3484	0.4140	0.2919	0.3507	0.4254
	ns=500	0.2961	0.3191	0.3421	0.2954	0.3216	0.3472

Table 2. Estimated Correlation Coefficient (100 Sample Sets)

	Min.	Mean	Max.
ns=10	-0.4667	0.4760	0.8667
ns=50	0.2767	0.5017	0.6261
ns=100	0.3665	0.5020	0.6319
ns=500	0.4564	0.5016	0.5630

For the comparison test, first, RBDO with the estimated standard deviations is carried out using 100 randomly generated data sets. Second, RBDO with the upper bounds of the input standard deviations obtained using 95% confidence level on the input model is carried out using the same data sets. Finally, PBDO is carried out using the estimated standard deviations. When RBDO is carried out, the MPP-based DRM is used for a more accurate result. Table 3 compares the three test results with the MCS result.

Table 3. Probability of Failure at Optimum Design (100 Sample Sets)

		P_{F2},%				P_{F3},%				Average Cost	No. of Design Failure
		Min.	Mean	Max.	No. of Const. Failure	Min.	Mean	Max.	No. of Const. Failure		
RBDO with Estimated	ns=10	0.1690	3.5447	23.6064	50	0.0074	3.9577	26.4990	51	-3.1920	59
	ns=50	0.4655	2.5461	6.1035	56	0.5294	2.5919	7.0129	54	-3.1507	64
	ns=100	0.7656	2.3012	4.2547	47	0.8114	2.3205	4.9048	46	-3.1201	54
	ns=500	1.5118	2.2199	3.1161	40	1.5065	2.2404	3.1172	41	-3.1211	45
RBDO with Upper Bound	ns=10	0.0000	0.3016	9.2656	3	0.0000	0.2228	10.9155	1	-1.7175	3
	ns=50	0.0408	0.8080	2.8207	4	0.0526	0.6693	2.8114	3	-2.6106	5
	ns=100	0.2205	1.0466	2.3313	2	0.2238	0.9173	2.3713	1	-2.7548	3
	ns=500	1.0264	1.6075	2.3745	1	0.9716	1.5398	2.2175	0	-2.9683	1
PBDO	ns=10	0.0000	0.9735	15.6695	10	0.0000	0.5674	10.3706	5	-2.3249	11
	ns=50	0.0093	0.3865	1.8642	0	0.0067	0.1723	0.9359	0	-2.2811	0
	ns=100	0.0398	0.3032	0.8663	0	0.0111	0.1261	0.3820	0	-2.2464	0
	ns=500	0.1180	0.2641	0.4942	0	0.0475	0.1134	0.1926	0	-2.2503	0

As shown in Table 3, regardless of the number of samples, RBDO with the estimated standard deviation shows about 50% probability of design failure, which means that the probability of failure of two constraints at the optimum design is larger than the target probability of failure (2.275%). In the table, "No. of Failure" means the number of cases out of 100 that the probability of failure for each constraint is larger than the target probability of failure, and "No. of Design Failure" means the number of cases that any constraint fails.

When the number of samples is 10 (ns=10), PBDO shows 11 design failures since there exist some cases in which the estimated standard deviations are too small compared to the true one, which is 0.3. As the number of samples increases, mean probability of failure for each constraint in PBDO decreases and is far from the target probability of failure, which results in no design failure. This means the optimum designs obtained using PBDO are not true optimum designs since the average cost of PBDO is much larger than the cost (-3.1270) obtained using the true input information. Hence, in PBDO, the increased number of samples does not help improve the design optimization.

On the other hand, RBDO with the upper bound of the input standard deviations consistently shows reliable optimum designs, which means that the number of design failure is about 5% or less, and the probability of failure at optimum design converges to the target probability of failure as the number of samples increases. In addition, average cost is much larger than the cost of PBDO while maintaining 5% or less design failure. Hence, RBDO with the upper bound of the input standard deviations provides a more stable and reliable optimum design than RBDO with the estimated standard deviations or PBDO. Furthermore, users can control the design optimization by changing the confidence level of the input standard deviations while others cannot.

7.2. M1A1 Abrams Tank Roadarm

The roadarm of the M1A1 tank [9] is used to compare MVDO and RBDO with confidence level on the input model for an engineering problem with insufficient data. The roadarm is modeled using 1572 eight-node isoparametric finite elements (SOLID45) and four beam elements (BEAM44) of ANSYS [30], as shown in Fig. 4, and is made of S4340 steel with Young's modulus $E=3.0\times10^7$ psi and Poisson's ratio $\nu=0.3$. The durability analysis of the roadarm is carried out using Durability and Reliability Analysis Workspace (DRAW) [31,32] to obtain the fatigue life contour as shown in Fig. 5. The fatigue lives at the critical nodes shown in Fig. 5 are chosen as the design constraints of the MVDO.

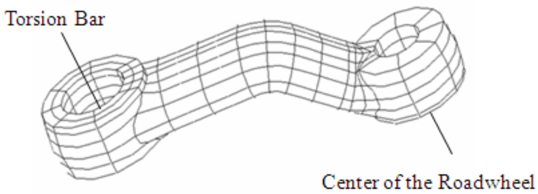

Figure 4. Finite Element Model of Roadarm

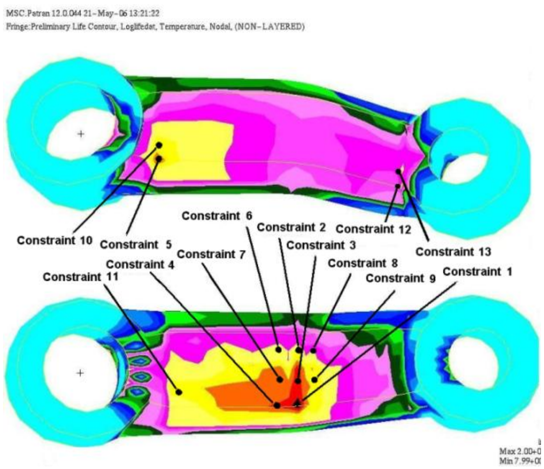

Figure 5. Fatigue Life Contour and Critical Nodes of Roadarm

The shape design variables are shown in Fig. 6. Eight shape design variables characterize four cross-sectional shapes of the roadarm. The widths (x_1-direction) of the cross-sectional shapes are defined by the design variables d_1, d_3, d_5, and d_7 at intersections 1, 2, 3, and 4, respectively, and the heights (x_3-direction) of the cross-sectional shapes are defined using the remaining four design variables. Eight shape design variables are listed in Table 4 and assumed to be independent random variables.

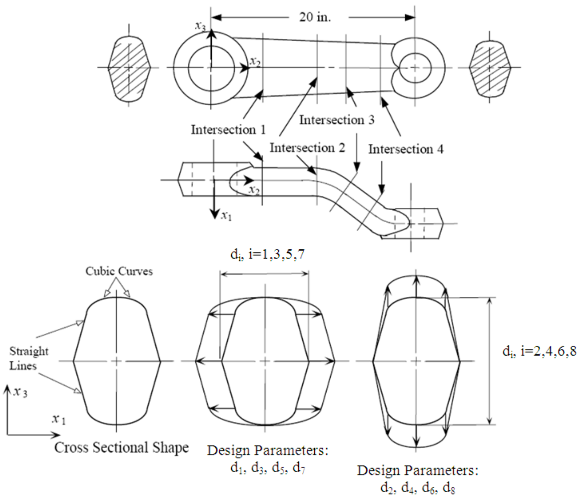

Figure 6. Shape Design Variables for Roadarm

Table 4. Random Variables and Fatigue Material Properties

Random Variables	Lower Bound $\mathbf{d}^L$	Initial Design $\mathbf{d}^0$	Upper Bound $\mathbf{d}^U$	Standard Deviation	Distribution Type
d_1	1.3500	1.7500	2.1500	0.0525	Gaussian
d_2	2.6496	3.2496	3.7496	0.0975	Gaussian
d_3	1.3500	1.7500	2.1500	0.0525	Gaussian
d_4	2.5703	3.1703	3.6703	0.0951	Gaussian
d_5	1.3563	1.7563	2.1563	0.0525	Gaussian
d_6	2.4377	3.0377	3.5377	0.0911	Gaussian
d_7	1.3517	1.7517	2.1517	0.0525	Gaussian
d_8	2.5085	2.9085	3.4085	0.0873	Gaussian

Fatigue Material Properties

Non-design Uncertainties	Mean	Standard Deviation	Distribution Type
Fatigue Strength Coefficient, σ_f'	177000	17700	Lognormal
Fatigue Strength Exponent, b	-0.0730	0.0073	Gaussian
Fatigue Ductility Coefficient, ε_f'	0.4100	0.0820	Lognormal
Fatigue Ductility Exponent, c	-0.6000	0.0600	Gaussian

For the input fatigue material properties, since the statistical information on S4340 steel other than its nominal value is not available, it is necessary to assume the statistical information on S4340 steel. Strain-Life relationship is usually given by the classical Coffin-Manson equation as [33]

$$\frac{\Delta\varepsilon}{2}=\frac{\Delta\varepsilon_e}{2}+\frac{\Delta\varepsilon_p}{2}=\frac{\sigma_f'}{E}(2N_f)^b+\varepsilon_f'(2N_f)^c \tag{46}$$

where σ_f' and b are the fatigue strength coefficient and exponent; ε_f' and c are the fatigue ductility coefficient and exponent; N_f is the fatigue initiation life; and E is the Young's modulus. It is known that σ_f', b, and ε_f', c are highly negatively correlated [18,20,21]. Furthermore, it is also known that σ_f' and ε_f' follow the lognormal distribution and b and c follow the Gaussian distribution. Hence, in this paper, it is assumed that σ_f' and b follow Gaussian copula with $\rho=-0.828$, and ε_f' and c follow Frank copula with $\tau=-0.906$. For the standard deviations of S4340 steel, 20% coefficients of variation (COV) for ε_f' and 10% COV for other fatigue material properties are assumed. The assumed statistical information of the fatigue material properties are also presented in Table 4.

To establish the design optimization for the roadarm, 30 samples are generated from the true marginal and joint distributions. Using 30 samples, the joint distributions are identified using the Bayesian method, and parameters for the identified distributions are quantified. Since the parameters are estimated using a limited number of samples (ns=30), it is not reliable to use the estimated parameters directly for the design optimization. Hence, they are treated as fuzzy variables for MVDO, or upper bounds of their standard deviations are used for RBDO with confidence level on the input model. The estimated mean (μ^e), standard deviation (σ^e), upper bound of the standard deviation (σ^{U}), correlation coefficient (τ^e), and identified copula using the Bayesian method are listed in Table 5.

Table 5. Estimated Parameters and Identified Copula

	σ'_f	b	ε'_f	c
μ^e	176423	-0.07297	0.4181	-0.6107
σ^e	13816	0.005746	0.06164	0.04969
σ^{U}	18574	0.007724	0.08286	0.06681
τ^e	−0.8050		−0.9264	
Copula	Gaussian		Frank	

The MVDO and RBDO with confidence level on the input model for the roadarm can be formulated to

$$\begin{aligned} &\text{minimize} \quad \mathrm{Cost}(\mathbf{d}) \\ &\text{subject to} \quad G_i(\mathbf{x}^*) \le 0,\ i=1,\cdots,13 \\ &\qquad \mathbf{d}^{\mathrm{L}} \le \mathbf{d} \le \mathbf{d}^{\mathrm{U}}, \mathbf{d} \in R^8 \ \text{and} \ \mathbf{X} \in R^{12} \end{aligned} \tag{47}$$

where

$$\begin{aligned} &\mathrm{Cost}(\mathbf{d}): \text{Weight of Roadarm} \\ &G_i(\mathbf{d}) = 1 - \frac{L(\mathbf{d})}{L_t},\ i = 1 \sim 13 \\ &L(\mathbf{d}): \text{Crack Initiation Fatigue Life,} \\ &L_t: \text{Crack Initiation Target Fatigue Life (=5 years)} \\ &\alpha_{t_i} = 0.02275 \text{ corresponding to } P_{F_i}^{\mathrm{Tar}} = 2.275\% \end{aligned} \tag{48}$$

and $\mathbf{x}^*$ is obtained from the inverse analysis given by Eq. (36) for the MVDO and is obtained from the MPP-based DRM for the RBDO with confidence level on the input model.

For comparison purposes, first, the MVDO is carried out using the estimated standard deviations for the fatigue material properties. Second, the RBDO with the estimated standard deviation and the upper bound of the standard deviation obtained using 95% confidence level on the input model is carried out. Finally, all these test results are compared with the results obtained from the RBDO with the true input statistical information. These results are shown in Table 6.

Table 6. Optimum Design Comparison

	Initial	DO*	MVDO	RBDO		
				UB**	Estimated	True
d_1	1.750	1.588	1.915	1.789	1.766	1.774
d_2	3.250	2.650	2.650	2.650	2.650	2.650
d_3	1.750	1.922	2.069	1.999	1.988	1.990
d_4	3.170	2.570	2.570	2.570	2.570	2.570
d_5	1.756	1.477	1.755	1.610	1.588	1.590
d_6	3.038	3.292	3.538	3.471	3.467	3.494
d_7	1.752	1.630	1.991	1.863	1.809	1.832
d_8	2.908	2.508	2.508	2.508	2.508	2.508
Cost	515.09	464.56	509.87	491.06	487.07	488.79

* deterministic optimum.
** upper bound obtained using 95% confidence level.

As shown in Table 6, the RBDO with the estimated parameters and identified copulas shows the unreliable optimum design compared with the cost obtained from the RBDO with the true input due to the underestimated standard deviations. Since MCS cannot be used for the benchmark test, the simple cost value comparison is used to determine whether or not the optimum design is more reliable. The cost of the MVDO is too large compared with the cost of the RBDO with the true input; thus, this optimum design is too conservative. However, the cost at the optimum design of the RBDO with 95%

confidence level on the input model is larger than that of the true optimum; hence, it is reliable but not too conservative because it is very close to the true optimum.

8. Discussions and Conclusion

In many industrial engineering applications, it is not easy to obtain accurate input statistical information due to the expensive experimental cost. In such a case, it could be unreliable to use the estimated parameters from insufficient data for the design optimization. To assure a reliable design for problems with correlated input and lack of statistical information, this paper proposes and compares the RBDO with confidence level on the input model and the PBDO or MVDO.

One mathematical example and one engineering example are used to compare two methods when insufficient data are available. The mathematical example shows that the PBDO could be unreliable when standard deviations are underestimated and the number of samples is very small; however, the optimum designs of PBDO are not true optimum when the number of samples is relatively large. On the other hand, the RBDO with confidence level on the input model yields reliable optimum designs in a consistent and stable manner. Furthermore, as the number of samples increases, the optimum design of the RBDO with confidence level on the input model converges to the optimum of the RBDO with the true input statistical model.

9. Acknowledgement

Research is supported by the Automotive Research Center, which is sponsored by the U.S. Army Tank Automotive Research, Development and Engineering Center (TARDEC).

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