Efficiency analysis of Reliability Based Design Optimization approaches for dependent input random variables

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1. Abstract

A deterministic optimization does not account for the uncertainties in the design variables and parameters. Modern competitive market demands have required the designers to introduce techniques for obtaining optimized designs that are also reliable. In the past twenty-five years, researchers have proposed a variety of methods to obtain optimum and reliable designs. These methods are addressed in Reliability-Based Design Optimization (RBDO). There are various types of RBDO approaches: Double-Loop methods, Decoupled methods and Single-Loop methods. This paper studies the efficiency of the various RBDO approaches applied to problems with dependent non-normal random input variables. Usually, the joint cumulative distribution function of this random vector is seldom available. In practice, it is further recognized that a general random vector could only be characterized reliably up to the marginal distributions and a measure of dependence between such as the popular linear correlation matrix. However, such limited information could not be enough to defined uniquely a general random vector.

First Order Reliability Method (FORM) is the most widely used method for reliability analysis. This method usually requires a iso-probabilistic transformation from the dependent non normal input random variables in the original space to standard normal random variables in the standard space. Since only marginal distributions and the linear correlation matrix are available to define the input random vector, the Nataf transformation is often the first choice. Recently, Nataf transformation has been considered from the copula viewpoint because it is the composition of two functions: the Gaussian copula and a linear transformation. The Gaussian copula accurately constructs a joint cumulative distribution function for several types of dependent random vectors when the marginal distributions and the linear correlation matrix are available. Since this information could be obtained from real data, Nataf transformation could be used in many practical RBDO applications. However, using a Gaussian copula and, therefore, a linear correlation matrix to model a general random vector might generate several pitfalls and difficulties.

A structural design example with dependent loads is provided in this paper to show the practical applicability of the Nataf transformation in RBDO. Several RBDO approaches are considered. The numerical efficiency and convergence are checked. The computational cost is assessed by the amount of optimization iterations and the total of performance functions evaluations.

2. Keywords: Structural Reliability, Reliability Based Design Optimization, Nataf Transformation, Copulas, Correlated random variables

3. Introduction

Design optimization has undergone a substantial progress. Commercial finite element codes have added optimization methods. However, most of these developments deals only with deterministic parameters. Uncertainties need to be considered in the design of any engineering system to assure reliability and quality. Traditional deterministic design methods have accounted for uncertainties through empirical safety factors. However, such safety factors do not provide a quantitative measure of the safety margin in design and are not quantitatively linked to the influence of different design variables and their uncertainties on the overall system performance. For a rational design to be made it is crucial to account for uncertain properties of material, loading and geometry as well as the mathematical model of the system. Also, any type of dependence or correlation between these uncertainties must be accounted for. The process of design optimization enhanced by the addition of reliability constraints is referred to as reliability-based design optimization (RBDO).

The aim of RBDO is to achieve optimal objective while ensuring adequate reliability. There are various RBDO methods. They are related to different formulations for the objective function and the reliability constraints. Objective function is a measure that we want to minimize and can be chosen between these types: weight of the structure, life-cycle cost of a product, probability of failure. Reliability constrains can include requirements about the probability of failure of individual components as well as the probability of failure of the entire system. However, the most widely used RBDO formulation contains an objective function of the type minimization of the weight subject to component level reliability constraints. This formulation is stated in the section 4. During the last twenty five years several RBDO approaches have been proposed. The main challenge of researchers is to obtain efficient and robust methods to avoid the high computational cost and convergence difficulties of the early methods. Several proposed methods are briefly described in section 5.

An important property of any RBDO method is generality, that is, it has to address a wide range of problems including analytical problems, structural problems and large scale engineering design problems. Also, an efficient RBDO method has to solve problems with several types of variables: deterministic design variables, random design variables, and other deterministic and random system variables. Frequently, in many engineering applications two input random variables are related. For example, material properties and fatigue properties are correlated [1]. Therefore, correlated input variables must be accounted in RBDO approaches and this paper studies their efficiency. If a RBDO method applies the First Order Reliability Method (FORM) to evaluate reliability constraints, correlated non normal input random variables in the original space are converted to uncorrelated standard normal random variables. Nataf transformation is the more widely implemented model to carry out this task. Recently, Nataf transformation has been reviewed from the copula viewpoint. Nataf transformation uses the Gaussian copula to obtain a joint distribution for correlated random variables if their marginal distributions and linear correlation matrix are known. Nataf transformation, therefore, inherits the advantages and drawbacks of the Gaussian copula. These subjects are studied in section 6. The classic ten bar truss problem shows the applicability of the Nataf transformation in RBDO in section 7. Results are verified by Importance Sampling Monte Carlo Simulation (MCS). Finally, section 8 includes the conclusions.

4. Formulation of the RBDO problem

A typical RBDO problem is formulated as

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}} f\left(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{P}}\right) \\ &s.t. \quad P_{fi} = P\left[g_i\left(\mathbf{d},\mathbf{X},\mathbf{P}\right)\leq 0\right]\leq P_{fi}^t, \quad i=1,\dots,n \\ &\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \qquad \boldsymbol{\mu}_{\mathbf{X}}^L \leq \boldsymbol{\mu}_{\mathbf{X}} \leq \boldsymbol{\mu}_{\mathbf{X}}^U \end{aligned} \tag{1}$$

where $\mathbf{d}\in R^k$ is the vector of deterministic design variables, $\mathbf{X}\in R^m$ is the vector of random design variables, $\mathbf{P}\in R^q$ is the vector of random parameters, $f(\cdot)$ is the objective function, n is the number of constraints, k is the number of deterministic design variables, m is the number or random design variables and q is the number of random parameters, P_{fi} is the probability of violating the *i*-th probabilistic constraint and P_{fi}^{t} is the target probability of failure for the *i*-th probabilistic constraint.

If the First Order Reliability Method (FORM) is used, as usually occurs in practical applications, the failure probability P_f of a probabilistic constraint is given as a function of the reliability index β, written as:

$$P_f \approx \mathbf{\Phi}(-\beta) \tag{2}$$

where $\mathbf{\Phi}(\cdot)$ is the standard Gaussian cumulated distribution function and β is the reliability index defined by Hasofer and Lind (1974), which is evaluated by solving the constrained optimization problem:

$$\begin{aligned} &\beta = \min\|\mathbf{u}\| = \left(\mathbf{u}^T \cdot \mathbf{u}\right)^{\frac{1}{2}} \\ &s.t.: G(\mathbf{d},\mathbf{u}) = 0 \end{aligned} \tag{3}$$

The solution of this optimization problem $\mathbf{u}^*$ is the minimum distance of a point $\mathbf{u}$ on the failure surface $G(\mathbf{d},\mathbf{u}) = 0$ from the origin of the standard normal space $\mathbf{U}$ and is called the Most Probable Point (MPP) or β -point, as $\beta = \|\mathbf{u}^*\|$. A probabilistic transformation $\mathbf{U} = T(\mathbf{X},\mathbf{P})$ from the original space of physical random variables $(\mathbf{X},\mathbf{P})$ to the normalized space $\mathbf{U}$ is needed. The image of a performance function $g_i(\mathbf{d},\mathbf{X},\mathbf{P})$ is $G_i(\mathbf{d},T(\mathbf{X},\mathbf{P})) = G_i(\mathbf{d},\mathbf{u})$. In subsequent sections this transformation will be explained.
The optimization process of Eq. (1) is carried out in the space of the design parameters $(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}})$. In parallel, the solution of the reliability problem of Eq. (3) is performed in the space of the random variables.
Traditional RBDO requires a double loop iteration procedure, where reliability analysis is carried out in the inner loop for each change in the design variables, in order to evaluate the reliability constraints. The computational time for this procedure is extremely high due to the multiplication of the number of iterations in both outer loop and reliability assessment loop, involving a very high number of mechanical analyses.

5. RBDO approaches

Due to the prohibitive computational effort of the traditional double loop RBDO method, researchers have developed several approaches to solve the numerical difficulties. Below, they are briefly described:

5.1 Double-loop Approaches

These RBDO formulations are based on improvements of the traditional double-loop approach by increasing the efficiency of the reliability analysis. Two approaches have been proposed to deal with probabilistic constraints in the double-loop formulation: Reliability Index Approach (RIA) and Performance Measure Approach (PMA).
RIA based RBDO is the traditional or classic RBDO formulation. The RBDO problem is solved in two spaces: the spaces of design variables, corresponding to a deterministic physical space and the space of Gaussian random variables, obtained by probabilistic transformation of the random physical variables. The RIA based RBDO problem is stated as:

$$\begin{aligned}&\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} f\left(\mathbf{d},\boldsymbol{\mu}_\mathbf{X},\boldsymbol{\mu}_\mathbf{P}\right)\\&s.t.\quad \beta_i\left(\mathbf{d},\mathbf{X},\mathbf{P}\right)\ge\beta_i^t,\quad i=1,\dots,n\\&\mathbf{d}^L\le\mathbf{d}\le\mathbf{d}^U,\qquad \boldsymbol{\mu}_\mathbf{X}^L\le\boldsymbol{\mu}_\mathbf{X}\le\boldsymbol{\mu}_\mathbf{X}^U\end{aligned}\tag{4}$$

where β_i is the reliability index of the *i*-th probabilistic constraint for the structure and β_i^t is the target or allowed reliability index. The calculation of the reliability index β_i involves solving the optimization problem stated as Eq.(3). Both, standard nonlinear constrained optimization methods and FORM methods like the Hasofer-Lind-Rackwitz-Fiesler (HLRF) method or the improved-HLRF method can be used.

The solution of this RBDO problem consists in solving the two nested optimization problems. For each new set of the design parameters, the reliability analysis is performed in order to get the new MPP, corresponding to a given reliability level. It is well established in the literature that RIA-RBDO converges slowly, or provides inaccurate results or even fails to converge due to the highly non linear transformations involved.

Lee and Kwak [2] and Tu *et al*. [3] proposed the use of a Performance Measure Approach instead of the widely used RIA. In PMA, inverse reliability analysis is performed to search for a point with the lowest positive performance function value on a hypersurface determined by the target reliability index β^t. Since the inverse reliability analysis is also performed iteratively, the reliability analysis and optimization loops are still nested.

The PMA-based RBDO problem is stated as

$$\begin{aligned}&\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} f\left(\mathbf{d},\boldsymbol{\mu}_\mathbf{X},\boldsymbol{\mu}_\mathbf{P}\right)\\&s.t.\quad G_{pi}=F_G^{-1}\left[\Phi\left(-\beta_{t_i}\right)\right]\ge 0,\quad i=1,\dots,n\\&\mathbf{d}^L\le\mathbf{d}\le\mathbf{d}^U,\qquad \boldsymbol{\mu}_\mathbf{X}^L\le\boldsymbol{\mu}_\mathbf{X}\le\boldsymbol{\mu}_\mathbf{X}^U\end{aligned}\tag{5}$$

where the performance measure G_p is obtained from the following reliability minimisation problem:

$$\begin{aligned}&G_p=\min_{\mathbf{u}} G\left(\mathbf{d},\mathbf{u}\right)\\&s.t.\quad \left\|\mathbf{u}\right\|=\beta_t\end{aligned}\tag{6}$$

The computational expense due to the nesting of design optimization and reliability analysis loops makes traditional RBDO impractical for large realistic problems. Various techniques have been proposed to improve the efficiency of RBDO. Some techniques improve the efficiency of reliability analysis in the double loop formulations. Other techniques decouple the design optimization and the reliability analysis. There are also techniques by which these design optimization and reliability analysis are carried out in one loop.

5.2 Improvements of reliability analysis in double-loop formulations.

PMA-based RBDO is shown to be more efficient and robust than RIA-based RBDO, because performance measure methods do not obtain the exact value of the probability of failure for each inner loop and this implies that computational cost decreases. However, several numerical examples using PMA show inefficiency and instability in the assessment of probabilistic constraints during the RBDO process. B.D. Youn *et al* [4], [5], [6] have carried out some PMA-based methods that are more efficient that Advanced Mean Value (AMV) method. These methods are Hybrid Mean Value (HMV) method [4], Enhanced Hybrid Mean Value (HMV+) method [5] and Enriched Performance Measure Approach (PMA+) [6] and are briefly described here. HMV is a combination of the Advance Mean Value and the Conjugate Mean

Value methods and provides accurate results for concave and convex nonlinear reliability constraints. However, although the HMV method performs well for convex or concave performance functions, it could fail to converge for highly nonlinear performance functions. An enhanced HMV method, named HMV+ was proposed by B.D. Youn *et al* [5]. This method improves numerical efficiency and stability substantially in reliability analysis for highly nonlinear performance functions. This method introduces an interpolation search to improve the inverse reliability MPP search. PMA+ is an enriched PMA enhancing numerical efficiency while maintaining stability in the RBDO process. PMA+ integrates three key ideas, in addition to the HMV+ method: launching RBDO at a deterministic optimum design, feasibility checks for probabilistic constraints and fast reliability analysis under the condition of design closeness.

5.3 Decoupled Methods or Sequential Methods.

In these RBDO approaches the optimization problem and the reliability assessment are decoupled and are carried out sequentially. The reliability constraints are replaced by equivalent deterministic (or pseudo-deterministic) constraints, involving some additional simplifications.

Du and Chen [7] proposed a sequential optimization and reliability assessment (SORA) method. This method employs a decoupled strategy where a series of cycles of optimization and reliability assessment is employed. In each cycle, design optimization and reliability assessment are decoupled from each other; no reliability assessment is required within the optimization and the reliability assessment in only conducted after the optimization. The key concept is to use the reliability information obtained in the previous cycle to shift the boundaries of the violated deterministic constraints (with low reliability) to the feasible region. Therefore, the design is improved from cycle to cycle and the computation efficiency is improved significantly.

5.4 Single loop approaches

These RBDO approaches collapse the optimization and the reliability problems within a single-loop dealing with both design and random variables. Both RIA- and PMA-based single-loop strategies have been developed. [8], [9], [10].

The single-loop single-vector (SLSV) approach [8], [11] provides the first attempt in a truly single-loop approach. It improves the RBDO computational efficiency by eliminating the inner reliability loops. However, it requires a probabilistic active set strategy for identifying the active constraints, which may hinder its practicality.

Other single-level RBDO algorithms have also been reported in Argarwal *et al*. [12], Kuschel and Rackwitz [9] and Streicher and Rackwitz [13]. These methods introduce the Karush-Kuhn-Tucker (KKT) optimality conditions at the optimums of the inner optimization loops as equality constraints in the outer design optimization loop. This helps to adopt a well-known strategy for effectiveness in optimization, i.e., satisfying the constraints only at the optimum and allowing the solution to be infeasible before convergence. However, these RBDO methods based on KKT conditions have a great drawback: the number of design variables increases and becomes the sum of the original design variables, the components of the MPP in standard normal space for each reliability constraints and the Lagrange multipliers for each optimization sub problem. This can increase the computational cost substantially, especially for practical problems with a large number of design variables and a large number of constraints. Furthermore, the approach in [9], [12] and [13] requires second-order derivatives which are computationally costly and difficult to calculate accurately.

Liang *et al* [14] have development a single-loop RBDO formulation. This method has a main advantage: it eliminates the repeated reliability loops without increasing the number of design variables or adding equality constraints. It does not require second-order derivatives. The KKT optimality conditions of the inner reliability loops are explicitly used to move from the standard normal $\mathbf{U}$ space to the original $\mathbf{X}$ space, where the inequality constraints of the outer design optimization loop are evaluated. It converts the probabilistic optimization formulation into a deterministic optimization formulation. This method estimates the MPP for each probabilistic constraint using gradient information from the previous iteration. It therefore, eliminates the reliability optimization loop of the conventional double-loop RBDO approach.

6 Applicability of RBDO approaches for dependent input variables

Generality refers here to applicability for different types of problems. A RBDO approach must solve different types of problems: analytical or mathematical problems, engineering problems represented analytically, structural problems which require calling to a FEA tool and large scale problems. If the number of design variables or probabilistic constraints is very high, an approximate or surrogate model could be obtained through response surface or adaptive response surface methods. Then, a RBDO approach could be applied to this approximate model.

A fundamental aspect is the type of random input variables allowed. This paper studies RBDO approaches applied to a wide range of problems with dependent input random variables. When FORM-based RBDO approaches are used, a probabilistic transformation from the non-normal correlated input random variables space $\mathbf{X}$ into uncorrelated standard normal variables at the normalized space $\mathbf{U}$ must be incorporated. Rosenblatt transformation [15] and the Nataf transformation [16] are the most representative transformations for correlated input variables. The Rosenblatt transformation requires the joint Cumulative Distribution Function (CDF) of the input random variables. This joint CDF is rarely available in practical applications. Therefore, Rosenblatt transformation can be used only if the joint CDF is given or input variables are independent. The Nataf transformation only requires the marginal CDFs and the linear correlation matrix of the input random variables. Unlike the joint CDF, this information is easily available in practical applications and because that Nataf transformation is selected.

Nataf transformation can be easily adapted for RBDO approaches. Here, Nataf transformation is implemented in three RBDO approaches: RIA-based double loop approach, PMA-based double loop approach and the well known decoupled approach named SORA early described. Single loop RBDO approaches might not be suited to address correlated input variables and are not considerate in this work.

Recently, the Nataf transformation has been viewed as a Gaussian copula by several researches [17], [18]. Consequently, we briefly describe the concept of copula and second we describe the Nataf transformation. Then, important remarks about the applicability of the Nataf transformation are exposed.

6.1 Theory of copulas

The theory of copulas is stated by Nelsen (1999) [19]: *"Copulas are functions that joint or couple multivariable distribution function to their one-dimensional marginal distribution functions. Alternatively, copulas are multivariable distribution functions whose one-dimensional margins are uniform on the interval [0 1]"*.

Using a copula, we can construct a multivariable distribution by specifying marginal univariate distributions, and then choose a copula to provide a dependence structure between variables. Bivariate distributions, as well as distributions in higher dimensions, are possible.

There are various measures of dependence to summarise the dependence structure between variables. Some of them are widely used: Pearson Rho or linear correlation coefficient, Spearman correlation coefficient, Kendall`s tau, and tail dependence coefficients. There are several types of copulas and they can group in classes or families. The more important copulas are: Perfect dependence and independence copulas, Gaussian copula, t-Student copula, Arquimedean copulas (Gumbel, Clayton and Frank copulas). Some of theses copulas can model only very limited types of dependence.

An important result about copulas is the Sklar's theorem (1959) that is stated as:
"Let $\mathbf{X} = (x_1, ..., x_n)$ *be a vector of random variables with a joint distribution* $F_{X_1 \ldots X_n}(x_1, ..., x_n)$ *and marginal distributions,* $F_{X_1}(x_1), \ldots, F_{X_n}(x_n)$. *There exists an n-dimensional copula* C *such that*

$$F_{X_1 \ldots X_n}(x_1, ..., x_n) = C(F_{X_1}(x_1), ..., F_{X_n}(x_n)) \tag{7}$$

If the marginal distributions $F_{X_i}(x_i)$ *are continuous, the copula C is unique; otherwise C is uniquely determined only on* $Range(F_1) \times \cdots \times Range(F_n)$.
On the other hand, consider a copula C and univariate function distributions. Then $F_{X_1 \ldots X_n}(x_1, ..., x_n)$ *as defined in Eq. (7) is a joint CDF with marginal distributions* $F_{X_1}(x_1), ..., F_{X_n}(x_n)$*".*
It is interesting to rewrite Eq. (7) for the copula itself:

$$C(\mathbf{u}) = F_{X_1 \ldots X_n}\left(F_{X_1}^{-1}(u_1), ..., F_{X_n}^{-1}(u_n)\right) \tag{8}$$

where $F_{X_i}^{-1}(\cdot)$ is the generalized inverse of $F_{X_i}(\cdot)$. Eq. (7) can be viewed as a theoretical tool to obtain the copula from a multivariable distribution function. This equation allows extracting a copula directly from a multivariable function. Here, we obtain the Gaussian copula from the multivariable normal distribution.

6.2 The Gaussian copula
The Gaussian copula is a link between a multivariate normal joint CDF and marginal CDFs.

$$C_{\Phi}(u_1, ..., u_n | \mathbf{P}') = \Phi_{\mathbf{P}'}\left(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n) | \mathbf{P}'\right), \quad \mathbf{u} \in I^n \tag{9}$$

where u_i can be any arbitrary marginal CDF $F_{X_i}(x_i)$ with values in $[0,1]$, that is, they can be normal or non-normal marginal CDF, $\mathbf{P}'$ is a linear correlation matrix, and $\Phi_{\mathbf{P}'}$ is CDF for a n-dimensional normal distribution with zero mean and linear correlation matrix $\mathbf{P}'$. The components of $\mathbf{P}'$ are the linear correlation coefficients or Pearson's Rho between the random variables $(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n))$.
We recall that the linear correlation coefficient between two correlated input variables (X, Y) is defined as

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}} \tag{10}$$

6.3 Nataf Transformation
The Nataf transformation transfers correlated input variables $\mathbf{X}$ with marginal CDF $F_{X_i}(x_i)$ and linear covariance matrix $\mathbf{P} = [\rho_{ij}]$ into independent standard normal variables $\mathbf{U}$ through two steps:
In the first step, the original random variables $\mathbf{X}$ are transformed into correlated standard normal variables $\mathbf{Y}$ with the linear correlation matrix $\mathbf{P}' = [\rho'_{ij}]$. The dependence structure between these standard normal variables $\mathbf{Y}$ is modelled by a Gaussian copula parameterised

with the matrix $\mathbf{P}'$. Therefore, the correlated standard normal variables are obtained using

$$y_i = \Phi^{-1}\left[F_{X_i}(x_i)\right], \quad i = 1,\dots,n \tag{11}$$

The hard task in this step is the computation of the components $\left[\rho'_{ij}\right]$ of the linear correlation matrix $\mathbf{P}'$. This requires solving this equation.

$$\rho_{ij} = \int_{-\infty}^{\infty} .. \int_{-\infty}^{\infty} \left(\frac{X_i - \mu X_i}{\sigma X_i}\right)\left(\frac{X_j - \mu X_j}{\sigma X_j}\right)\phi\left(y_i, y_j; \rho'_{ij}\right) dy_i dy_j \tag{12}$$

where $\left(\dfrac{X_i - \mu_{X_i}}{\sigma_{X_i}}\right)$ is the normalized random variable for X_i, $\phi\left(y_i, y_j; \rho'_{ij}\right)$ is the standard bivariate normal Probability Density Function (PDF) and ρ_{ij} is the linear correlation coefficient between X_i and X_j. The exact solution of Eq. (12) requires a great computational effort. Liu and Der Kiureghian [20] have solved this equation and have estimated the linear correlation coefficients ρ'_{ij} from ρ_{ij} for different types of input variables. They provided 49 empirical formulas with ρ'_{ij} in terms of ρ_{ij} for 10 different probability distributions. Also, they provide the range of values for correlation coefficients where Nataf transformation may be applicable for combinations of these probability distributions.

In the second step, the correlated standard normal variables $\mathbf{Y}$ with their multivariable joint CDF known from the first step are transformed to uncorrelated standard normal variables $\mathbf{U}$ through the linear transformation stated as:

$$\mathbf{U} = \mathbf{L}_0^{-1}\mathbf{Y} \tag{13}$$

where $\mathbf{L_0}$ is the lower triangular matrix obtained from Cholesky decomposition of $\mathbf{P}'$.

Therefore, Nataf Transformation is equivalent to choice a Gaussian copula for the joint distribution of the input random variables $\mathbf{X}$. This copula is parameterized by a linear correlation matrix $\mathbf{P}'$ is such a way that the joint distribution of $\mathbf{X}$ has the given linear correlation matrix $\mathbf{P}$. Eq. (12) allows obtaining matrix $\mathbf{P}'$ from matrix $\mathbf{P}$, but it has some limitations.

6.4 Applicability of the Nataf transformation

R. Lebrun and A. Dutfoy [17] have described some potential pitfalls of using the Nataf transformation and the linear correlation as a measure of dependence. The choice of a specific family of copulas determines the choice of a specific parameter as measure of dependence. Here, the choice of a Gaussian copula implies the choice of a linear correlation like dependence measure. Linear correlation is invariant under linear transformations, but not under general transformations. This can produce wrong results when it is considered outside the multivariate normal distributions. Gaussian copula does not allow taking into account any positive tail dependence, because this copula is asymptotically independent in both upper and lower tails. This means, no matter what high linear correlation exists; there will be no tail dependence from a Gaussian copula. Upper and lower tail dependence coefficients are dependence measures in extreme values of the random variables. These are important measures when we are interested in the evaluation of a probability of failure for a system with dependent random variables, because failure usually occurs when these random variables are in their extreme values into the failure region. Therefore, the use of Gaussian copula might lead to a highly underestimated probability of failure.

Linear correlation coefficient is a well studied concept; unfortunately, it is only a suitable dependence measure in a special class of distributions, i.e. elliptical distributions. This class

includes the normal distribution and mixtures of normal distribution. It is well known that beyond this class the use of linear correlation leads to a number of pitfalls and fallacies.
Let be (X_1, X_2) two random variables with marginal CDFs $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, there exist values in $[-1,1]$ that cannot be reached by the linear correlation coefficient between X_1 y X_2, whatever the copula we choose: these values are not compatible with the chosen marginal distributions and the linear correlation coefficient ρ_{12} takes values in an interval [$\rho_{12}^{\min}, \rho_{12}^{\max}$] included in $[-1,1]$. This drawback is emphasized in Nataf transformation and the consequence is the impossibility to solve Eq. (12) for values of ρ_{ij} close to -1 and 1 and for some probability distributions [20].
Important properties of linear correlation matrix must be checked. This matrix must be definite positive and symmetric with all its diagonal elements equal to 1 and the others in [-1, 1]. Therefore, if Nataf transformation is used, these properties must be verified for both linear correlation matrix $\mathbf{P}$ and $\mathbf{P}'$. Sometimes, this is an important difficulty, especially for high dimension correlation matrix.
Yoojeong Noh *et al* [18] have also studied the applicability of Gaussian copula. They show that Gaussian copulas can construct the exact joint PDF in the two-dimensional case with one normal and one lognormal input random variables and can accurately estimate the joint CDF in the two dimensional case of correlated lognormal input variables if they are positively correlated or independent. The Gaussian copula may also accurately estimate the lognormal distribution with a small coefficient of variation because the shape of lognormal distribution with small coefficient of variation is very similar to the normal distribution. However it can not accurately model multivariate distributions with non-normal margins whose shapes are rather different from the normal distribution. They study the applicability of Gaussian copula for a two dimensional case of correlated input random variables with exponential distributions. They show the difficulties to model a joint exponential CDF by a Gaussian copula.
These difficulties involved in the construction of a joint CDF for dependent input random variables might be solved by using a different class of copulas with its specific dependence measures to parameter the dependence structure between the random variables. Further investigation must be carried out about dependence modelling between random variables and its applicability to RBDO problems. However, when experimental data are not available and the information about input random variables is only the marginal distributions and the linear correlation matrix, Nataf transformation can be considered the unique solution for structural reliability applications. When there is an experimental sample, several copulas could be fitted to the data and simple graphical tools and numerical techniques could be used to select an appropriate model.
The next section studies the application of several RBDO approaches to a structural example with correlated input variables. Different probability distributions are considered. The results are verified trough Importance Sampling MCS for the active constraints.

7 Structural example: ten bars truss

The ten bar truss structure shown in Fig. 1 is widely used by RBDO community. The weight of the ten bar truss is minimized subject to stress, displacements and bucking constraints. The structures is supporting correlated random loads, the mean value of the first one is $P_1 = 100kN$ applied in the node 1 and the mean value of the second one is $P_2 = 50kN$ applies in the nodes 2 and 4. The bars are manufactured in steel, with circular sections. Since the horizontal, vertical and diagonal members are cut from three different steel rods, their cross-sectional areas A_1, A_2 y A_3, respectively, are considered to be random design variables. Random variables are summarized in Table 1. Random input loads are assumed correlated. Standard deviations are assumed constant through the RBDO process.

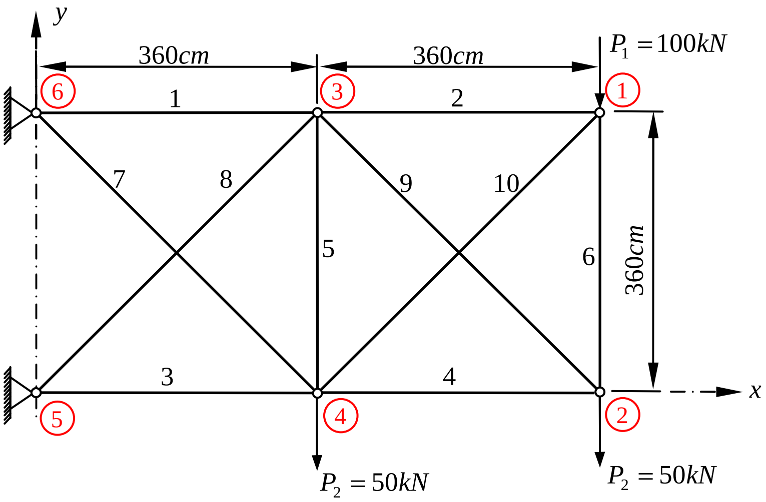

Figure 1. Ten bar Truss

In this RBDO example, the volume of the structure is minimized subject to 11 reliability constraints corresponded to state limit functions. Design variables are constrained by lateral bounds. The RBDO formulation is stated as:

$$\begin{aligned} & Min \quad V(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\mu}_{\mathbf{P}}) \\ & s.t. \quad \beta_i \geq \beta_i^t = 3.7 \qquad i = 1,..,11. \\ & \qquad 5\,\mathrm{cm}^2 \leq \mu_{X_i} \leq 75\,\mathrm{cm}^2 \qquad i = 1,2,3. \end{aligned} \tag{14}$$

where the reliability constraints are the subsequent state limit functions:

Displacement constraint: Vertical displacement of the node 2 (q_V^2) is limited:

$$g_1(\mathbf{d}, \mathbf{X}, \mathbf{P}) = 1 - \frac{|q_V^2(\mathbf{d}, \mathbf{X}, \mathbf{P})|}{q^a} \tag{15}$$

where the allowed displacement is $q^a = 3.5cm$

Stress constraints: Maximum stress of the element is limited:

$$g_i(\mathbf{d},\mathbf{X},\mathbf{P}) = 1 - \frac{|\sigma_{i-1}(\mathbf{d},\mathbf{X},\mathbf{P})|}{\sigma_{i-1}^a(\mathbf{d},\mathbf{X},\mathbf{P})} \qquad i = 2,...,11 \tag{16}$$

where σ^a is the maximum stress allowed and, in the case of bars in compression, it regards the buckling of the bar and takes the value of the Euler's critical stress, that is stated as:

$$\sigma^{cr} = \frac{\pi^2 E I_{min}}{l^2 A} \tag{17}$$

where E is the elasticity module, $I_{\min}$ is the minimum moment of inertia of the cross section, l is the length of the bar and A is the cross area. A target reliability index $\beta_t = 3.7$ was fixed for all the constraints. This reliability index value corresponds to a failure probability of $1.0780 \cdot 10^{-4}$.

Three RBDO approaches are applied to the problem: RIA-based double loop method (RBDO-RIA), PMA-based double loop method and SORA. Since inverse reliability analysis was computed by using the enhanced Hybrid Mean Value (HMV+) method the last two RBDO approaches are named RBDO-HMV+ and SORA-HMV+, respectively. A linear elastic analysis is assumed to evaluate the structural response. All sensitivity calculations were carried out analytically

Table 1: Random Variables in ten bar truss

Random Variable	Description	Type of Distribution	Initial Mean	Standard Deviation	Design Variable
X_1	A_1	LN	20.0 cm^2	1.0 cm^2	μ_{X_1}
X_2	A_2	LN	20.0 cm^2	1.0 cm^2	μ_{X2}
X_3	A_3	LN	20.0 cm^2	1.0 cm^2	μ_{X3}
X_4	E	LN	21000.0 kN/cm^2	1000 kN/cm^2	-
X_5	σ^a	LN	21.0 kN/cm^2	20 kN/cm^2	-
X_6	P_1	G	100.0 kN	20 kN	-
X_7	P_2	LN	50.0 kN	2.5 kN	-

Case *a*): $P_1 \sim G(100,20)$ and $P_2 \sim LN(50,2.5)$, where G means Gumbel or Extreme value Type I distribution and *LN*, lognormal distribution. The linear correlation coefficient ρ between these variables can not take values in the entire range [-1, 1], because if ρ is closed to 1 or -1, the linear correlation matrix is not definite positive and, therefore, Nataf transformation can not be applied.

The results obtained from RBDO-HMV+ (Table 2) and RBDO-SORA (Table 3) approaches are practically the same. As shown in the tables, the optimum cross areas and volumes for the structure significantly depend on the linear correlation coefficient ρ.

The RBDO-RIA approach does not converge. The nonlinearity involved in mapping random variables from the original space $\mathbf{X}$ to the standard normal space $\mathbf{U}$, especially with the Gumbel distribution, is the reason of this lack of convergence. For the sake of a comparative analysis the convergence tolerance is set as 10^{-4} for all the iterative algorithms. The computational efficiency of the RBDO methods is expressed by the number of optimization iterations and Limit State Functions (LSF) or performance functions evaluations. Last three columns in Table 3 contain the LSF evaluations for the sequential optimization cycle, the LSF evaluations for the reliability assessment cycle and the total number of LSF evaluations, respectively.

Result are verified by Importance Sampling Monte Carlo Simulation (MCS) for the active constrains (g_2, g_7 and g_8) and the error in terms of index reliability is below 0.4%.

Table 2: Case *a* - RBDO-HMV+ results

ρ	A_1, cm^2	A_2, cm^2	A_3, cm^2	Volume, cm^3	Opt. Iters.	LSF Eval
0.95	59.6390	24.0069	64.2111	234056.719	23	760
0.80	59.4516	24.1064	63.9198	233264.972	23	762
0.60	59.2051	24.2341	63.5339	232215.226	23	766
0.40	58.9622	24.3564	63.1509	231172.851	23	787
0.20	58.7232	24.4737	62.7734	230143.686	23	809
0.00	58.4882	24.5866	62.3980	229121.177	23	813
-0.20	58.2572	24.6955	62.0264	228109.321	23	856
-0.40	58.0303	24.8006	61.6589	227109.179	23	859
-0.60	57.8077	24.9021	61.2959	226121.851	23	860
-0.80	57.5894	25.0005	60.9379	225148.456	23	864
-0.95	57.4286	25.0722	60.6728	224428.215	23	874

Table 3: Case *a* - SORA-HMV+ results

ρ	A_1, cm^2	A_2, cm^2	A_3, cm^2	Volume, cm^3	Opt. Iters.	LSF Eval OPT	LSF Eval REL	LSF Eval SUM
0.95	59.6390	24.0069	64.2111	234056.783	4	396	132	528
0.80	59.4516	24.1065	63.9198	233265.035	4	396	132	528
0.60	59.2051	24.2341	63.5340	232215.290	4	396	132	528
0.40	58.9622	24.3564	63.1510	231172.917	4	396	135	531
0.20	58.7232	24.4737	62.7735	230143.722	4	396	140	536
0.00	58.4882	24.5866	62.3980	229121.208	4	396	141	537
-0.20	58.2571	24.6955	62.0264	228109.348	4	396	149	545
-0.40	58.0303	24.8005	61.6589	227109.201	4	440	149	589
-0.60	57.8076	24.9021	61.2959	226121.870	4	473	149	622
-0.80	57.5894	25.0005	60.9379	225148.471	4	506	149	655
-0.95	57.4286	25.0722	60.6728	224428.229	4	429	148	577

Case *b*): $P_1 \sim LN(100,20)$ and $P_2 \sim LN(50,2.5)$. Now, the two correlated load are lognormal distributed. The rest of random variables remain without changes. In this case the three methods (RBDO-RIA, RBDO-HMV+ and RBDO-SORA) converge to the same results (Tables 4, 5 and 6). Results are verified by Importance Sampling MCS for the active constrains (g_2 , g_7 and g_8) and the error in terms of index reliability is below 0.4%.

Table 4: Case *b* - RBDO-RIA results

ρ	A_1, cm^2	A_2, cm^2	A_3, cm^2	Volume, cm^3	Opt. Iters.	LSF Eval
0.95	57.0238	22.2165	61.8842	224258.5545	20	1310
0.80	56.8419	22.3186	61.6070	223504.9957	20	1310
0.60	56.6003	22.4493	61.2363	222495.5781	20	1313
0.40	56.3601	22.5744	60.8646	221482.1702	6	393
0.20	56.1218	22.6944	60.4903	220462.3006	6	388
0.00	55.8856	22.8097	60.1178	219445.9717	5	355
-0.20	55.6519	22.9205	59.7478	218435.1344	5	352
-0.40	55.4213	23.0273	59.3769	217423.7647	5	337
-0.60	55.1939	23.1303	59.0079	216418.2776	6	396
-0.80	54.9703	23.2297	58.6416	215421.1287	9	581
-0.95	54.8053	23.3020	58.3692	214680.2668	13	807

Table 5: Case *b* - RBDO-HMV+ results

ρ	A_1, cm^2	A_2, cm^2	A_3, cm^2	Volume, cm^3	Opt. Iters.	LSF Eval.
0.95	57.0238	22.2165	61.8842	224258.5293	9	297
0.80	56.8419	22.3186	61.6069	223504.8911	9	297
0.60	56.6003	22.4493	61.2361	222495.2383	9	297
0.40	56.3601	22.5744	60.8643	221481.4061	9	300
0.20	56.1216	22.6944	60.4917	220464.9750	9	301
0.00	55.8854	22.8096	60.1188	219447.7224	9	306
-0.20	55.6517	22.9205	59.7463	218431.6546	9	310
-0.40	55.4210	23.0272	59.3769	217423.2526	9	318
-0.60	55.1936	23.1302	59.0079	216417.6939	9	320
-0.80	54.9700	23.2296	58.6415	215420.5287	9	323
-0.95	54.8050	23.3019	58.3691	214679.6956	10	360

Table 6: Case *b* - SORA-HMV+ results

ρ	A_1, cm^2	A_2, cm^2	A_3, cm^2	Volume, cm^3	Opt. Iters.	LSF Eval OPT	LSF Eval REL	LSF Eval SUM
0.95	57.0238	22.2165	61.8842	224258.5290	5	363	165	528
0.80	56.8419	22.3186	61.6069	223504.8909	5	363	165	528
0.60	56.6003	22.4493	61.2361	222495.2382	5	363	165	528
0.40	56.3601	22.5744	60.8643	221481.4063	5	473	165	638
0.20	56.1216	22.6944	60.4917	220464.9754	5	473	165	638
0.00	55.8854	22.8096	60.1188	219447.7230	5	473	169	642
-0.20	55.6517	22.9205	59.7463	218431.6554	5	484	169	653
-0.40	55.4210	23.0272	59.3769	217423.2534	5	583	175	758
-0.60	55.1936	23.1302	59.0079	216417.6949	5	583	177	760
-0.80	54.9700	23.2296	58.6415	215420.5299	5	583	178	761
-0.95	54.8050	23.3019	58.3692	214679.7027	5	572	178	750

The results for the two cases show that the number of LSF evaluations increases when the correlation coefficient decreases from 0.95 to -0.95. In case *a* SORA-HMV+ is more efficient than RBDO-HMV+, computationally speaking. However, RBDO-HMV+ is more efficient than SORA-HMV+ in case *b*. From this structural example, we can conclude that correlation can be easily considered in the RBDO of structural problems. When an estimation of the correlation coefficient is not available or when linear correlation is time variant, we could take the higher values of design variables for any value of ρ . That is, in case *b)*, 57.0238, 23.3019 and 61.8842 cm^2 for the cross-sectional areas A_1 , A_2 y A_3, respectively.

8. Conclusions

The efficiency of three RBDO methods for dependent input variables has been presented. These methods were coded by the first author, basing on previous literature. Active constraints strategy and "warm up" strategy (set as starting point in the current MPP search for each reliability constraint, the MPP obtained from the last optimization cycle) were not development in the implementation of the RBDO methods for the sake of comparison between the methods. Therefore, the efficiency of these methods could be further improved with little programming effort.

Since only marginal CDFs and matrix correlation of the random input variables are known in practical applications, Nataf transformation is practically the unique choice in most RBDO problems to transform correlated random input variables from the original space to the standard uncorrelated normal space.

Nataf transformation has been studied as the composition of a Gaussian copula and a linear transformation. Advantages and drawbacks derived from using Nataf transformation have been described. Nevertheless, Nataf transformation works well in most real problems because probabilistic distributions of input random variables usually are normal or similar to normal and their coefficients of variation are low. Another advantage is that a wide range of correlation coefficients is covered by Nataf transformation. In this paper, a structural example shows that Nataf transformation is a valid tool for structural applications in RBDO. Results are verified by Importance Sampling MCS and the errors are very little.

The review of the Nataf transformation from the copula viewpoint opens several questions to the RBDO community and further investigation about the applicability of classes of copulas different to the Gaussian copula in RBDO.

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