Efficiency analysis of Reliability Based Design Optimization approaches for dependent non-normal random vectors

Luis Celorrio
Eduardo Martínez de Pisón
Carmen Bao

Department of Mechanical Engineering
University of La Rioja
Logroño (La Rioja)
SPAIN

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OUTLINE

- RBDO PROBLEM
- RBDO APROACHES
- ISOPROBABILISTIC TRANSFORMATIONS
- NATAF TRANSFORMATION
- COMPUTATION DRAWBACKS OF NATAF TRANSFORMATION
- ADVANTAGES OF NATAF TRANSFORMATION
- THEORY OF COPULAS
- GAUSSIAN COPULA
- NATAF TRANSFORMATION FROM THE NATAF VIEWPOINT
- POTENCIAL PITFALLS OF USING NATAF TRANSFORMATION
- POTENCIAL PITFALLS OF USING LINEAR CORRELATION
- STRUCTURAL EXAMPLE
- CONCLUSIONS

RBDO PROBLEM

The RBDO problem is written as:

$$\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}} f\left(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\mu}_{\mathbf{P}}\right)$$

$$s.t. \quad P_{fi} = P\left[g_i\left(\mathbf{d},\mathbf{X},\mathbf{P}\right)\leq 0\right]\leq P_{fi}^{t}, \quad i=1,\dots,n$$

$$\mathbf{d}^{L}\leq \mathbf{d}\leq \mathbf{d}^{U}, \qquad \boldsymbol{\mu}_{\mathbf{X}}^{L}\leq \boldsymbol{\mu}_{\mathbf{X}}\leq \boldsymbol{\mu}_{\mathbf{X}}^{U}$$

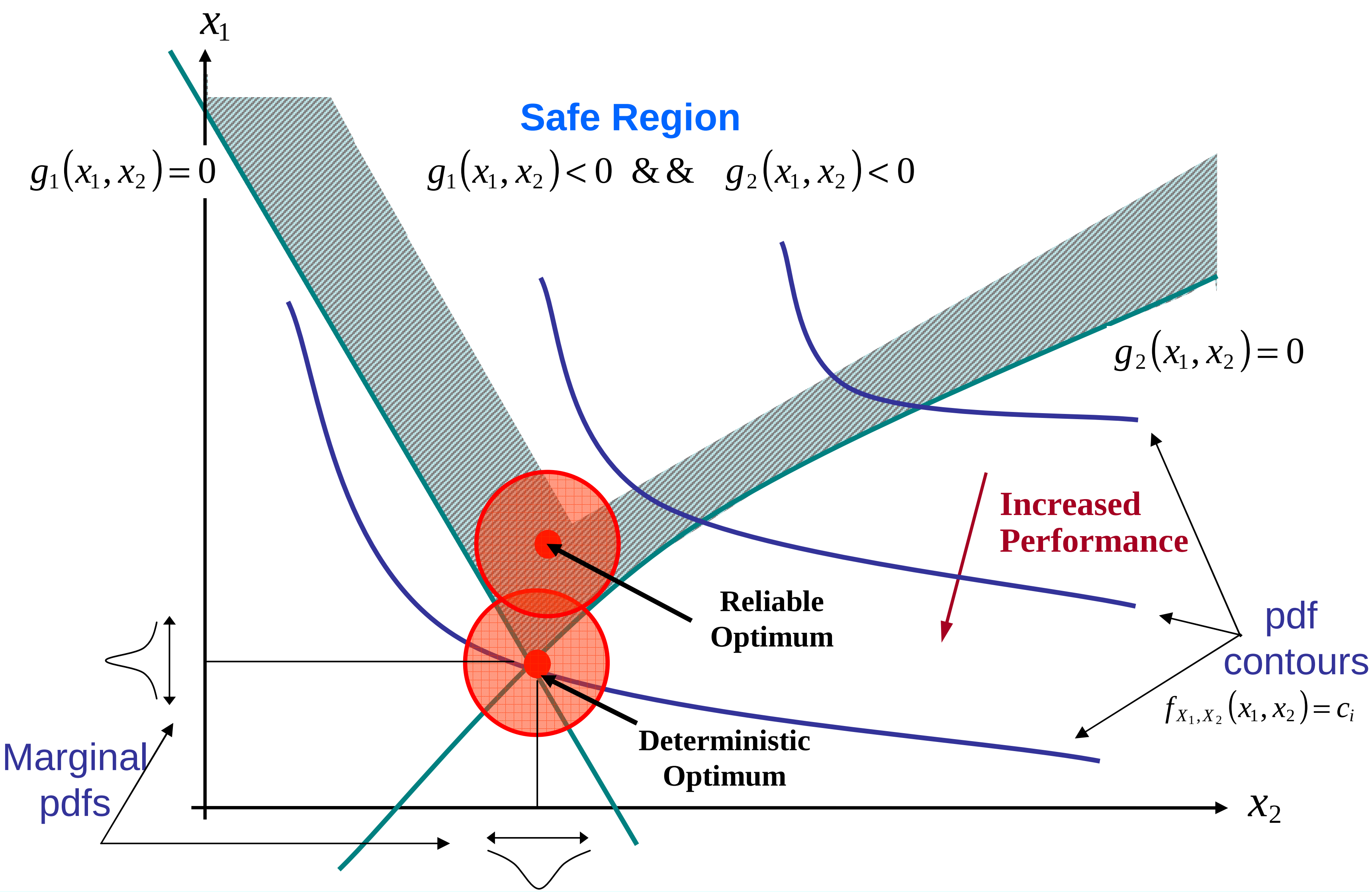
where:

$\mathbf{d}\in R^{k}$ **:** vector of deterministic design variables

$\mathbf{X}\in R^{m}$: vector of random design variables

$\mathbf{P}\in R^{q}$: vector of random parameters

- Single objective function
- Component level probabilistic constraints
- $g_i\left(\mathbf{d},\mathbf{X},\mathbf{P}\right)\leq 0$ Indicates Failure



x_1
Safe Region
$g_1(x_1,x_2)=0$
$g_1(x_1,x_2)<0 \;\&\&\; g_2(x_1,x_2)<0$
$g_2(x_1,x_2)=0$
Increased Performance
Reliable Optimum
Deterministic Optimum
pdf contours
$f_{X_1,X_2}(x_1,x_2)=c_i$
Marginal pdfs
x_2

RBDO APPROACHES

- Double loop approaches:
  - RIA-based double loop RBDO (classical formulation)
  - PMA-based double loop RBDO

    Several PMA algorithms: HMV, HMV+, etc.
- Decoupled (or sequential) approaches; e.g. SORA.
- Single loop approaches; e.g. SLSV, methods based on to collapse KKT conditions of reliability loop as a constraint of the outer design loop.

FORM-based RBDO approaches requires an Isoprobabilistic Transformation. The most representative are:

Rosenblatt Transformation:

- Can be applied when the joint CDF of the random vector is available.
- This rarely occurs in real applications

Nataf Transformation:

- Usually, only marginal CDFs and the linear correlation matrix $\mathbf{P}$ of the random vector are known. The elements of $\mathbf{P}$ are noted ρ_{ij}
- This transformation allows to map the space of the input random variables $\mathbf{X}$ into the space of independent standard normal variables $\mathbf{U}$.

NATAF TRANSFORMATION

Nataf Transformation T is the composition of two functions $T = T_2 \circ T_1$ such that

$$T_1 : \mathbf{X} \mapsto \mathbf{Y} = \begin{pmatrix} \Phi^{-1}(F_1(X_1)) \\ \Phi^{-1}(F_2(X_2)) \\ \vdots \\ \Phi^{-1}(F_n(X_n)) \end{pmatrix} \qquad T_2 : \mathbf{Y} \mapsto \mathbf{U} = \Gamma \mathbf{Y}$$

- $\mathbf{Y}$ is a gaussian vector with a correlation matrix $\mathbf{P}'$ and with standard normal marginal distributions.
- Γ is any square-root matrix of $\mathbf{P}'^{-1}$, often Cholesky factor of $\mathbf{P}'^{-1}$
- Φ is CDF of the standard normal distribution

➡ $\mathbf{U}=T(\mathbf{X})$, is a vector of independent standard normal variables.

- The computation of coefficients ρ'_{ij} of the matrix **P'** might be very difficult and tedious. The integral equation below must be solved.

$$\rho_{ij} = \int_{-\infty}^{\infty} .. \int_{-\infty}^{\infty} \left(\frac{X_i - \mu_i}{\sigma_i} \right) \left(\frac{X_j - \mu_j}{\sigma_j} \right) \phi_2 \left(y_i, y_j; \rho'_{ij} \right) dy_i dy_j$$

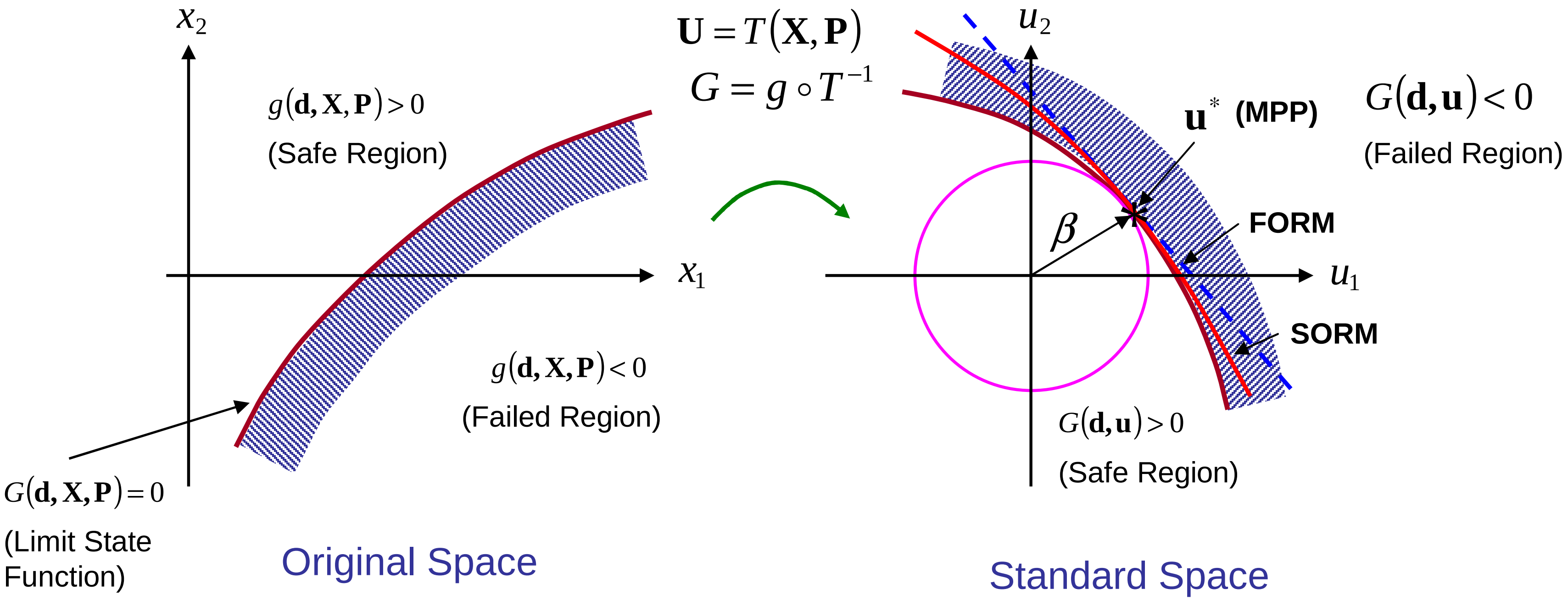
where ϕ_2 is the bivariate standard normal probability density function with correlation ρ'_{ij}

- There is no guarantee that the resulting matrix **P'** will be symmetric definite positive.

ADVANTAGES OF NATAF TRANSFORMATION

- Nataf Transformation maps the physical or original space where $\mathbf{X}$ takes its values into the stardard space where $\mathbf{U}$ take its values. It is the main aim.
- First Order Reliability Method (FORM) or Second Order Reliability Method (SORM) are used to identify the "most probable point" (MPP) $\mathbf{u}^*$

Nataf Transformation



Original Space

Standard Space

THEORY OF COPULAS

Copula is a general way of formulating a multivariable distribution.

Copulas are tools for modelling dependence of several random variables.

Main idea: a simple transformation can be made of each marginal variable to obtain its marginal cumulative distribution function. Each marginal CDF has a uniform distribution on the interval [0,1]. Then, the copula expresses the dependence structure of this uniforms distributions and .

Selection of a Copula must be done through fitting texts to experimental data.

Sklar's theorem (1959):

Let $\mathbf{X} = (x_1, ..., x_2)$ be a vector of random variables with a joint distribution $F_{X_1,..,X_n}(x_1, ..., x_2)$ and marginal distributions $F_{X_1}(x_1), ..., F_{X_n}(x_n)$

There exists an n-dimensional copula C such that

$$F_{X_1...X_n}(x_1, ..., x_n) = C(F_{X_1}(x_1), ..., F_{X_n}(x_n))$$

If the marginal distributions $F_{X_i}(x_i)$ are continuous, the copula C is unique.

$$\left.\begin{array}{l} F_{X_1...X_n}(x_1, ..., x_n) \\ F_{X_1}(x_1), ..., F_{X_n}(x_n) \end{array}\right\} \Rightarrow C = F\left(F_{X_1}^{-1}(u_1), ..., F_{X_n}^{-n}(u_n)\right)$$

for any $\mathbf{u} = (u_1, ..., u_n)$ in $[0,1]^n$

where $F_{X_i}^{-1}$ is the inverse of F_{X_i}

Multivarite normal distribution leads to the Gaussian copula.
Gaussian copula is a link between a multivariate normal joint CDF and marginal CDFs, parameterized by a linear correlation matrix $\mathbf{P}'$

$$C_\Phi\left(u_1,\ldots,u_n \middle| \mathbf{P}'\right) = \Phi_{\mathbf{P}'}\left(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n) \middle| \mathbf{P}'\right), \quad \mathbf{u} \in I^n$$

There are 3 elements:

Margins: Arbitrary marginal CDFs $u_i = F_{X_i}(x_i)$

Measure of Dependence: Linear Correlation Matrix $\mathbf{P}'$ between normal r.v. $\left(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n)\right)$

The Copula: A joint normal CDF $\Phi_{\mathbf{P}'}$

NATAF TRANSFORMATION

Two steps: $T = T_2 \circ T_1$

1.- Gaussian copula: $T_1 : \mathbf{X} \mapsto \mathbf{Y}$

2.- Linear Transformation (Rosenblatt Transformation)

$$T_2 : \mathbf{Y} \mapsto \mathbf{U} = \Gamma \mathbf{Y}$$

Nataf Transformation inherits the features of Gaussian Copula

Other copulas could fit the data more accurately

It is possible to chose different copulas that lead to joint distributions with the same rest two elements: marginal CDFs and linear correlation matrix The bidimensional case allows more choices of different copulas than the generalized n-dimensional case with marginal CDFs and linear correlation matrix known.

Dependence structure could not be suited

The choice of the Gaussian copula implies a very specific form of dependence structure and a choice of dependence measure (linear correlation coefficient) to summarize the dependence structure, which might not be suited for the problem considered.

Tail dependence can not be considered with Gaussian copula

The Coefficients of Upper and Lower Tail Dependence are measures of association that summarize the dependence structure in the extreme values of the variables. However Gaussian copula can not take into account any positive tail dependence.

Description of the dependence structure

A copula describes fully the dependence structure between random variables. However, a measure of association is by no way a full representation of the dependence structure.

Gaussian copula summarizes the dependence structure with a linear correlation or the classic Pearson Rho coefficient.

The linear correlation coefficient between random variables (*X,Y*) is defined as:

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}}$$

It is a dependence measure useful only for elliptical distributions

A linear correlation describe fully a multivariable elliptical distribution like a multivariable normal distribution.

It is non-invariant by a nonlinear marginal transformation

If g_1 and g_2 are strictly increasing funtions, we have :

$$\rho(X,Y) \neq \rho(g_1(X), g_2(Y))$$

Rank Correlation coefficients like Spearman Rho and Kendall Tau are invariant by a nonlinear marginal transformation.

The linear correlation coefficient matrix must be symmetrical definite positive matrix:

Both $\mathbf{P}$ and $\mathbf{P}'$ must be definite positive matrix. This premise might not occur especially for high dimensional matrix and/or when correlation coefficients are close to -1 or 1.

MATRIX $\mathbf{P}$ FOR THE STRUCTURAL EXAMPLE:

$$\mathbf{P} = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \rho \\ & & & & & \rho & 1 \end{bmatrix}$$

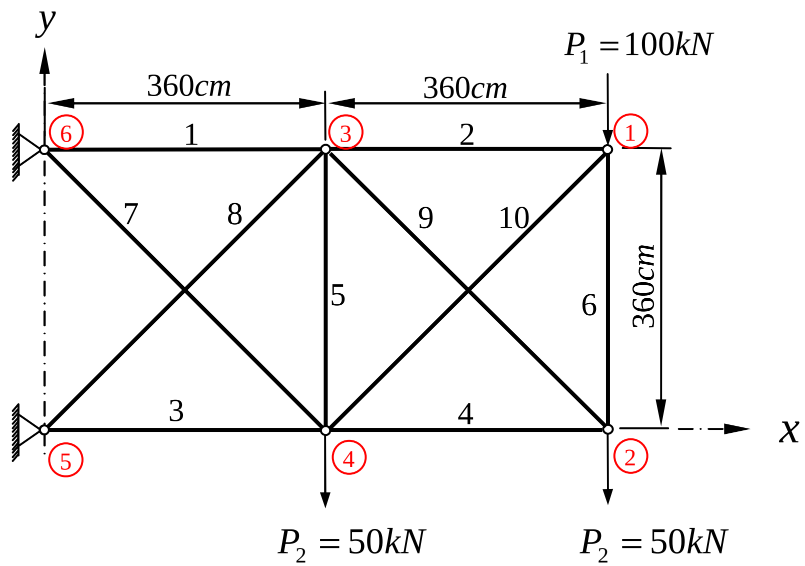
Where ρ is the linear correlation coefficient between the loads P_1 and P_2.

RBDO problem formulation

$$Min \quad V(\mathbf{d}, \boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\mu}_{\mathbf{P}})$$

$$s.t. \quad \beta_i \geq \beta_i^t = 3.7 \qquad i = 1,\ldots,11.$$

$$5\,\mathrm{cm}^2 \leq \mu_{X_i} \leq 75\,\mathrm{cm}^2 \qquad i = 1,2,3.$$



Reliability Constraints

Displacement constraint

$$g_1(\mathbf{d}, \mathbf{X}, \mathbf{P}) = 1 - \frac{\left| q_V^2(\mathbf{d}, \mathbf{X}, \mathbf{P}) \right|}{q^a}$$

$q^a = 3.5cm$ = *displacement allowed*

Stress constraints

$$g_i(\mathbf{d}, \mathbf{X}, \mathbf{P}) = 1 - \frac{\left| \sigma_{i-1}(\mathbf{d}, \mathbf{X}, \mathbf{P}) \right|}{\sigma_{i-1}^a(\mathbf{d}, \mathbf{X}, \mathbf{P})} \qquad i = 2,\ldots,11$$

σ^a = *stress allowed*

Buckling is regarded for compression bars through Euler's critical stress

Random Variables- Case a) P_1 ~ Gumbel and P_2 ~ LogNormal

Random Variable	Description	Type of Distribution	Initial Mean	Standard Deviation	Design Variable
X_1	A_1	LN	20.0 cm^2	1.0 cm^2	μ_{X_1}
X_2	A_2	LN	20.0 cm^2	1.0 cm^2	μ_{X_2}
X_3	A_3	LN	20.0 cm^2	1.0 cm^2	μ_{X_3}
X_4	E	LN	21000.0 kN/cm^2	1000 kN/cm^2	-
X_5	σ^a	LN	21.0 kN/cm^2	20 kN/cm^2	-
X_6	P_1	G	100.0 kN	20 kN	-
X_7	P_2	LN	50.0 kN	2.5 kN	-

Design Variables

μ_{X_1} Mean value of A_1 Cross-Sectional area of the horizontal bars

μ_{X_2} Mean value of A_2 Cross-Sectional area of the vertical bars

μ_{X_3} Mean value of A_3 Cross-Sectional area of the diagonal bars

RESULTS - Case a) P_1 ~ Gumbel and P_2 ~ LogNormal

RBDO-RIA Does not Converge

RBDO-HMV+

ρ	A_1, cm^2	A_2, cm^2	A_3, cm^2	Volume, cm^3	Opt. Iters.	LSF Eval
0.95	59.6390	24.0069	64.2111	234056.719	23	760
0.80	59.4516	24.1064	63.9198	233264.972	23	762
0.60	59.2051	24.2341	63.5339	232215.226	23	766
0.40	58.9622	24.3564	63.1509	231172.851	23	787
0.20	58.7232	24.4737	62.7734	230143.686	23	809
0.00	58.4882	24.5866	62.3980	229121.177	23	813
-0.20	58.2572	24.6955	62.0264	228109.321	23	856
-0.40	58.0303	24.8006	61.6589	227109.179	23	859
-0.60	57.8077	24.9021	61.2959	226121.851	23	860
-0.80	57.5894	25.0005	60.9379	225148.456	23	864
-0.95	57.4286	25.0722	60.6728	224428.215	23	874

RESULTS - Case a) P_1 ~ Gumbel and P_2 ~ LogNormal

SORA-HMV+

ρ	A_1, cm^2	A_2, cm^2	A_3, cm^2	Volume, cm^3	Opt. Iters.	LSF Eval OPT	LSF Eval REL	LSF Eval SUM
0.95	59.6390	24.0069	64.2111	234056.783	4	396	132	528
0.80	59.4516	24.1065	63.9198	233265.035	4	396	132	528
0.60	59.2051	24.2341	63.5340	232215.290	4	396	132	528
0.40	58.9622	24.3564	63.1510	231172.917	4	396	135	531
0.20	58.7232	24.4737	62.7735	230143.722	4	396	140	536
0.00	58.4882	24.5866	62.3980	229121.208	4	396	141	537
-0.20	58.2571	24.6955	62.0264	228109.348	4	396	149	545
-0.40	58.0303	24.8005	61.6589	227109.201	4	440	149	589
-0.60	57.8076	24.9021	61.2959	226121.870	4	473	149	622
-0.80	57.5894	25.0005	60.9379	225148.471	4	506	149	655
-0.95	57.4286	25.0722	60.6728	224428.229	4	429	148	577

STRUCTURAL EXAMPLE

CONCLUSIONS- Case a)

- RBDO-RIA does not converge because Natat Transformation involved a high non linearity for the Gumbel distribution.
- Optimum design values are practically the same for RBDO-HMV+ and SORA-HMV+ for all the range of ρ.
- SORA-HMV+ is more efficient than RBDO-HMV+ because it requires less Limit State Evaluations.
- Values of ρ close to 1 and -1 are not possible because linear matrix correlation is not positive defined and the Nataf Transformation is invalid.

Random Variables- Case b) P_1 and P_2 ~ LogNormal

Random Variable	Description	Type of Distribution	Initial Mean	Standard Deviation	Design Variable
X_1	A_1	LN	20.0 cm^2	1.0 cm^2	μ_{X_1}
X_2	A_2	LN	20.0 cm^2	1.0 cm^2	μ_{X_2}
X_3	A_3	LN	20.0 cm^2	1.0 cm^2	μ_{X_3}
X_4	E	LN	21000.0 kN/cm^2	1000 kN/cm^2	-
X_5	σ^a	LN	21.0 kN/cm^2	20 kN/cm^2	-
X_6	P_1	LN	100.0 kN	20 kN	-
X_7	P_2	LN	50.0 kN	2.5 kN	-

Design Variables

μ_{X_1} Mean value of A_1 Cross-Sectional area of the horizontal bars

μ_{X_2} Mean value of A_2 Cross-Sectional area of the vertical bars

μ_{X_3} Mean value of A_3 Cross-Sectional area of the diagonal bars

RESULTS - Case b) P_1 and P_2 ~ LogNormal

RBDO-RIA

ρ	A_1, cm^2	A_2, cm^2	A_3, cm^2	Volume, cm^3	Opt. Iters.	LSF Eval
0.95	57.0238	22.2165	61.8842	224258.5545	20	1310
0.80	56.8419	22.3186	61.6070	223504.9957	20	1310
0.60	56.6003	22.4493	61.2363	222495.5781	20	1313
0.40	56.3601	22.5744	60.8646	221482.1702	6	393
0.20	56.1218	22.6944	60.4903	220462.3006	6	388
0.00	55.8856	22.8097	60.1178	219445.9717	5	355
-0.20	55.6519	22.9205	59.7478	218435.1344	5	352
-0.40	55.4213	23.0273	59.3769	217423.7647	5	337
-0.60	55.1939	23.1303	59.0079	216418.2776	6	396
-0.80	54.9703	23.2297	58.6416	215421.1287	9	581
-0.95	54.8053	23.3020	58.3692	214680.2668	13	807

RESULTS - Case b) P_1 and P_2 ~ LogNormal

RBDO-HMV+

ρ	A_1, cm^2	A_2, cm^2	A_3, cm^2	Volume, cm^3	Opt. Iters.	LSF Eval.
0.95	57.0238	22.2165	61.8842	224258.5293	9	297
0.80	56.8419	22.3186	61.6069	223504.8911	9	297
0.60	56.6003	22.4493	61.2361	222495.2383	9	297
0.40	56.3601	22.5744	60.8643	221481.4061	9	300
0.20	56.1216	22.6944	60.4917	220464.9750	9	301
0.00	55.8854	22.8096	60.1188	219447.7224	9	306
-0.20	55.6517	22.9205	59.7463	218431.6546	9	310
-0.40	55.4210	23.0272	59.3769	217423.2526	9	318
-0.60	55.1936	23.1302	59.0079	216417.6939	9	320
-0.80	54.9700	23.2296	58.6415	215420.5287	9	323
-0.95	54.8050	23.3019	58.3691	214679.6956	10	360

RESULTS - Case b) P_1 and P_2 ~ LogNormal

SORA-HMV+

ρ	A_1, cm^2	A_2, cm^2	A_3, cm^2	Volume, cm^3	Opt. Iters.	LSF Eval OPT	LSF Eval REL	LSF Eval SUM
0.95	57.0238	22.2165	61.8842	224258.5290	5	363	165	528
0.80	56.8419	22.3186	61.6069	223504.8909	5	363	165	528
0.60	56.6003	22.4493	61.2361	222495.2382	5	363	165	528
0.40	56.3601	22.5744	60.8643	221481.4063	5	473	165	638
0.20	56.1216	22.6944	60.4917	220464.9754	5	473	165	638
0.00	55.8854	22.8096	60.1188	219447.7230	5	473	169	642
-0.20	55.6517	22.9205	59.7463	218431.6554	5	484	169	653
-0.40	55.4210	23.0272	59.3769	217423.2534	5	583	175	758
-0.60	55.1936	23.1302	59.0079	216417.6949	5	583	177	760
-0.80	54.9700	23.2296	58.6415	215420.5299	5	583	178	761
-0.95	54.8050	23.3019	58.3692	214679.7027	5	572	178	750

CONCLUSIONS- Case b)

- RBDO-RIA converge. However RBDO-HMV+ and SORAM-HMV+ are more effcient.
- Optimum design values are practically the same for the three methods.
- RBDO-HMV+ is more efficient than SORA-HMV+.
  LSF evaluations in RBDO-HMV+ are practically the half of LSF evaluations in SORA-HMV+ for any value of ρ
- Values of ρ close to 1 and -1 are not possible because linear matrix correlation is not positive defined and the Nataf Transformation is invalid.

CONCLUSIONS

- The efficiency of RBDO approaches for dependent input variables has been presented.

- Nataf transformation is applied in structural reliability when only the marginals CDFs and linear correlation matrix of input random variables are known.

- Nataf transformation is the composition of a Gaussian copula and a linear transformation and, therefore, inherits the advantages and drawbacks of the Gaussian copula.

- A structural example shows that Nataf transformation is a valid tool for structural applications in RBDO. Two dependent loads are considered and the computational effort of each RBDO approach is recorded by the number of Limit State Evaluations.

- Further investigation about the applicability of other type of copulas in RBDO problems might carry out when experimental sample data from dependent variables are available.

Thank you for your kindly attention

Luis Celorrio Barragué
luis.celorrio@unirioja.es
www.unirioja.es
Spain