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Reliability-based optimization of structures using probability and convex set mixed models

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1. Abstract
In certain circumstances of the safety assessment and the reliability-based design optimization of structures, the probability and convex set mixed models may be suitably used for the uncertainty description. Based on the probabilistic and convex set mixed model, this paper presents a mathematical definition of reliability index for measuring the safety of structures. The optimization problem is then mathematically formulated and converted into more tractable one. Moreover, the double-loop optimization problem is transformed into an approximate single-loop minimization problem using the linearization-based technique, which further facilitates efficient solution of the design problem. Numerical examples demonstrate the validity of the proposed formulation as well as the efficiency of the presented numerical techniques.
2. Keywords: structural optimization; reliability; probability; convex set; mixed model

3. Introduction
Along with the ever increasing computational power, the past two decades has seen a rapid development of structural optimization in both theories and industrial applications. In particular, the design optimization problem incorporating various uncertainties has been intensively studied. Among other non-deterministic optimal design formulations, the reliability-based design optimization provides an effective tool for seeking the best designs against structural failures in presence of system variations. Basically, the uncertainty models employed by a typical structural reliability analysis can be classified into two categories: the probabilistic model and non-probabilistic models. As the most mature uncertainty model, the probabilistic model describes the stochastic parameters and structural responses with random fields or discrete random variables that have certain statistical distribution characteristics. The probabilistic model has been successfully used in many real-life engineering applications for structural reliability-based design optimization (RBDO) [1,2] as well as robust design optimization [3,4]. In practical applications, the probabilistic distribution type and the corresponding statistical parameters of inputs are usually extracted from a sufficient amount of measured data or assumed on the basis of engineering experiences.
A meaningful probabilistic reliability analysis relies on availability of precise description of the statistical characteristics, particularly, the tail distribution of the random inputs. However, these data cannot be accurately extracted in some circumstances due to limited number of samples. As illustrated by Elishakoff [5], a small error in constructing the probabilistic density for input quantities may give rise to misleading prediction of the probabilistic reliability. This means that the traditional probabilistic approaches might be questionable to deal with those problems involving information-incomplete or inherently non-probabilistic uncertainties. Consequently, non-probabilistic models have also been developed as alternative models for describing uncertainty with incomplete statistical information [6].
This paper aims to provide a method to incorporate simultaneously randomness and uncertain-but-bounded uncertainties into the design optimization problem. To achieve this goal, a mathematical definition of structural mixed reliability index based on probabilistic model and convex set is first proposed. Then, a nested optimization model for reliability-based structural design problems with constraints on such mixed reliability indices is presented. To demonstrate the applicability of the proposed model and the efficiency of the numerical techniques, three pure mathematical or engineering design examples are presented.

2. Reliability-based design under mixed model of probability and convex sets
2.1. Pure probabilistic description
In the conventional probabilistic framework, the uncertainties are modelled as random variables with certain distribution characteristics. Let $\mathbf{x}=\{x_1, x_2, \mathbf{K}, x_m\}^{\mathrm{T}}$ denotes the vector of random variables, the structural failure probability can be given as

$$P_{\mathrm{f}} = \Pr\left[G(\mathbf{x}) \leq 0\right] = \int \mathbf{K} \int_{G(\mathbf{x})\leq 0} p_{\mathrm{x}}(\mathbf{x}) dx_1 \, \mathbf{K} \, dx_m \tag{1}$$

where $\Pr[\cdot]$ denotes the probability, $G(\mathbf{x})$ is a system performance function and $G(\mathbf{x}) \le 0$ defines the failure event, $p_{\mathrm{x}}(\mathbf{x})$ is the joint probability density function, which is usually approximated using measured data sets of the system parameters. The accuracy of the approximated $p_{\mathrm{x}}(\mathbf{x})$ is limited by the total number of available samples. For solving the multi-variate integral in (1), there are many available techniques, such as Rosenblatt's transformation (Rosenblatt 1952), for transforming the m-variate distribution $p_{\mathrm{x}}(\mathbf{x})$ into independent ones $p_j\left(x_j\right)\left(j=1,2,\mathbf{K},m\right)$.

2.2. Probability and convex set mixed model description

A practical engineering structure may exhibit both probabilistic uncertainties and bounded uncertainties. In such a case, the uncertain variables involved in the design problem can be classified into random variables and uncertain-but-bounded variables described by multi-ellipsoid convex sets. They are respectively denoted by $\mathbf{x}=\{x_1,x_2,\mathbf{K},x_m\}^{\mathrm{T}}$ and $\mathbf{y}=\{y_1,y_2,\mathbf{K},y_n\}^{\mathrm{T}}$, which are expressed as

$$\mathbf{x} \sim \left\{p_j\left(x_j\right), \quad j=1,2,\mathbf{K},m\right\}, \tag{2}$$

$$\mathbf{y} \in \mathbf{E} = \left\{\boldsymbol{\delta} : \boldsymbol{\delta}^{\mathrm{T}}\mathbf{W}\boldsymbol{\delta} \le \varepsilon^2\right\}, \tag{3}$$

where $p_j\left(x_j\right)$ is the probabilistic density function for the random variables x_j, $\mathbf{E}$ is the ellipsoid convex set defining the variation range of $\mathbf{y}$.

2.3 Reliability-based design optimization under mixed models

A structural optimization problem aims to seek the best design that satisfies certain structural behaviour requirements. Under the mixed model, the structural behaviours can be expressed as the performance functions of the design variables $\mathbf{d}$, the normalized probabilistic variables $\mathbf{u}$ and the normalized uncertain-but-bounded variables $\mathbf{v}$, namely $g\left(\mathbf{d},\mathbf{u},\mathbf{v}\right)$. It should be noted that the design variables can be also defined as the mean values or the nominal values of the uncertain variables. In the present paper, the reliability-based design optimization problem under the probability and convex set mixed model is mathematically formulated as

$$\begin{aligned} \min_{\mathbf{d}} \quad & f\left(\mathbf{d}\right) \\ \text{s.t.} \quad & a\left(\mathbf{d}\right) \ge 0 \\ & \mathbf{d}_{\mathrm{L}} \le \mathbf{d} \le \mathbf{d}_{\mathrm{U}} \end{aligned} \tag{4}$$

where

$$\begin{aligned} a\left(\mathbf{d}\right) = \min_{\mathbf{u},\mathbf{v}} \quad & g\left(\mathbf{d},\mathbf{u},\mathbf{v}\right) \\ \text{s.t.} \quad & \mathbf{u}^{\mathrm{T}}\mathbf{u} = \underline{b}_{\mathrm{m}}^{2} \\ & \mathbf{v}^{\mathrm{T}}\mathbf{v} \le 1 \end{aligned} \tag{5}$$

3. Solution strategy

The structural optimization problem incorporating reliability constraints under mixed modelling of probabilistic randomness and convex models presents a challenging problem with nested optimization. While a traditional nested double-loop approach is available, we proposed a linearization-based approach in this study to reduce the computational cost.

3.1. Nested double-loop approach

In the reliability-based optimization problem under the mixed model, the inner-loop of the target performance evaluation is embedded in the outer-loop for the overall design optimization. A direct double-loop procedure can be resorted for solving the above nested problem (4) and (5).

Though the nested double-loop approach is applicable, it still requires prohibitively lengthy calculations. Since the inner-loop evaluation of target performance needs many function evaluations of performance functions, and each iteration of the outer-loop optimization consists of an execution of the inner-loop minimization, the total number of function evaluations is usually very high.

3.2. Linearization-based approach

Various techniques have been developed to decouple the nested optimization problem involved in the conventional RBDO. In a sequential optimization strategy [8-10], the inner-loop and the outer-loop are treated sequentially and thus the optimum design is obtained by solving a sequence of sub-programming problems. In most practical circumstances, the variability of the uncertain-but-bounded variables is relatively small or moderate. Therefore, it is reasonable to assume that the performance functions are monotonic with respect to these quantities within their variation bounds. An iteration scheme for solving the optimum $\left(\mathbf{u}^*, \mathbf{v}^*\right)$ is presented in the following. Denoting the approximate solution of (5) in the k-th iteration by $\left(\mathbf{u}^{(k)}, \mathbf{v}^{(k)}\right)$, using the partial derivatives of the performance function expressed by

$$\mathbf{G}_{\mathbf{u}^{(k)}} = \left.\frac{\partial g}{\partial \mathbf{u}}\right|_{\mathbf{d}^{(k)},\mathbf{u}^{(k)},\mathbf{v}^{(k)}}, \mathbf{G}_{\mathbf{v}^{(k)}} = \left.\frac{\partial g}{\partial \mathbf{v}}\right|_{\mathbf{d}^{(k)},\mathbf{u}^{(k)},\mathbf{v}^{(k)}}, \tag{6}$$

a heuristic scheme for updating $\left(\mathbf{u}^*, \mathbf{v}^*\right)$ corresponding to the j-th reliability constraint would be

$$\left(\mathbf{u}^{(k+1)}, \mathbf{v}^{(k+1)}\right) = -\left(\frac{\underline{b}_{\mathrm{m}} \mathbf{G}_{\mathbf{u}^{(k)}}}{\sqrt{\mathbf{G}^{\mathrm{T}}_{\mathbf{u}^{(k)}} \mathbf{G}_{\mathbf{u}^{(k)}}}}, \frac{\mathbf{G}_{\mathbf{v}^{(k)}}}{\sqrt{\mathbf{G}^{\mathrm{T}}_{\mathbf{v}^{(k)}} \mathbf{G}_{\mathbf{v}^{(k)}}}}\right). \tag{7}$$

4. Numerical examples

4.1. Minimization of a mathematical function under reliability constraints

The first example considers minimization of an explicit performance function under reliability constraints. Two normally distributed random variables (denoted by x_1 and x_2) and two uncertain parameters (denoted by y_1 and y_2) bounded by an ellipsoid model are taken into account in the problem. The optimization problem is expressed as

$$\begin{aligned} \min_{\mathbf{d}} \quad & f(\mathbf{d}) = (d_1+3)^2 + (d_2+3)^2 \\ \text{s.t.} \quad & b_{\mathrm{m}}\left[g_1(\mathbf{x},\mathbf{y}) \ge 0\right] \ge \underline{b}_{\mathrm{m}}, \\ & b_{\mathrm{m}}\left[g_2(\mathbf{x},\mathbf{y}) \ge 0\right] \ge \underline{b}_{\mathrm{m}}, \\ & 0.01 \le d_1 \le 10,\ 0.01 \le d_1 \le 10, \end{aligned} \tag{8}$$

in which

$$\begin{aligned} & g_1(\mathbf{x},\mathbf{y}) = x_1(x_2+y_1) - y_2, \\ & g_2(\mathbf{x},\mathbf{y}) = x_1 - (x_2+y_1)^2 y_2, \end{aligned} \tag{9}$$

where the design variables are $\mathbf{d} = \{d_1, d_2\}^{\mathrm{T}}$, with d_1 and d_2 representing the mean values of x_1 and x_2, respectively. The coefficients of variation for x_1 and x_2 are both 0.03. Another two variables y_1 and y_2 are described by an ellipsoid model expressed by $\mathbf{y} = \{y_1, y_2\}^{\mathrm{T}} \in \mathbf{E} \equiv \left\{\mathbf{y} \middle| (\mathbf{y}-\hat{\mathbf{y}})^{\mathrm{T}} \mathbf{W}_{\mathrm{y}} (\mathbf{y}-\hat{\mathbf{y}}) \le 0.5^2\right\}$, where the nominal values $\hat{\mathbf{y}} = \{\hat{y}_1, \hat{y}_2\}^{\mathrm{T}} = \{0.25, 2\}^{\mathrm{T}}$ and $\mathbf{W}_{\mathrm{y}} = diag(4,1)$. The required reliability is $\underline{b}_{\mathrm{m}} = 3.0$.

For initial values of the design variables $d_1^{(0)} = d_2^{(0)} = 5$, the optimal solutions are listed in Table 1. The proposed linearization-based approach results in the identical optimal solutions as the nested double-loop approach. However, the linearization-based approach is much more efficient since it avoids the iteration of inner-loops. The iteration history plotted in Figure 1 shows a steady decrease of objective function as well as a stable convergence. For testing the dependency of the optimal solutions upon the initial guesses, three different initial values of design variables $d_i^{(0)} = 1,3,8$ $(i = 1,2)$ are also fed into the optimizer. From the iteration histories shown in Figure 1, it can be seen that the iterations converge to the same optimum, though the efficiency of the linearization-based approach is dependent on the initial design point.

Table 1: Solutions for the mathematic example

	Linearization-based approach	Nested double-loop approach
Objective	55.9733	55.9733
optimal design (d_1, d_2)	(3.5045, 0.6966)	(3.5045, 0.6966)
Nominal value $(\bar{x}_1, \bar{x}_2, \hat{y}_1, \hat{y}_2)$	(3.5045, 0.6966. 0.25, 2)	(3.5045, 0.6966. 0.25, 2)
$(x_1^*, x_2^*, y_1^*, y_2^*)$ for g_1	(3.2750, 0.6536, 0.0366, 2.2606)	(3.2752, 0.6536, 0.0365, 2.2600)
$(x_1^*, x_2^*, y_1^*, y_2^*)$ for g_2	(3.2901, 0.7426, 0.4699, 2.2380)	(3.2892, 0.7425, 0.4699, 2.2377)

Number of iterations for outer-loop	32	31
Total number of performance function evaluations	64	558

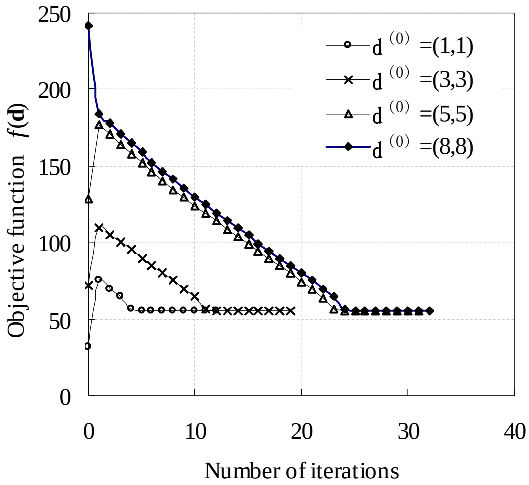

Figure 1: Iteration histories of the optimization with different initial design points

4.2. Reliability-based optimization of a ten-bar truss structure

Figure 2 shows a planar ten-bar truss structure, which is to be optimized for minimum weight. The horizontal and vertical bar members have a length of $L = 360$. The mass density of the material is $r = 0.1$. Two external loads P are applied to node 2 and node 4. A constraint $U \le 2.0$ is imposed on the vertical displacement of node 2. The bar cross-sectional areas $A_i\left(i = 1, 2, \mathbf{K}, 10\right)$ and the Young's modulus E of the material are Gaussian normal random variables, whereas the external load P is an uncertain-but-bounded variable.

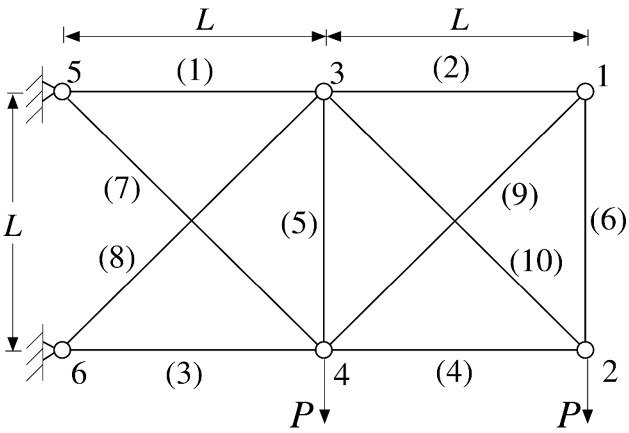

Figure 2: The ten-bar truss structure

The mean values of the member section areas $\bar{A}_i\left(i = 1, 2, \mathbf{K}, 10\right)$ are taken as design variables, with lower bounds $\underline{d}_i = 0.1$ and initial values $d_i^{(0)} = 40.0\left(i = 1, 2, \mathbf{K}, 10\right)$. The target reliability index is set as $\underline{b}_m = 3.0$.

For comparison's purpose, the deterministic optimization based on nominal values, the reliability-based optimization in the pure probabilistic framework (RBDO) and the worst-case scenario approach were also run, wherein all the uncertainties are assumed to have a Gaussian normal distribution with the coefficient of variation being 0.05 and the probabilistic reliability is required to be 3.0.

The numerical results for all the cases are listed in Table 2. For these optimal designs, the corresponding reliability indices evaluated using the mixed model parameters are also given in the last row of the table. The deterministic optimization presents a design with the least structural weight, though the reliability requirement is not accounted for. The RBDO is rather effective if all the probabilistic data of uncertainties are available. However, when the probabilistic and convex set mixed model is concerned, the RBDO solution has a reliability index of $b_{\mathrm{m}} = 1.50$ and thus also violates the reliability constraint. In the design obtained by the present method, a reliability index $b_{\mathrm{m}} = 3.00$ is achieved. The iteration history of the structural design problem plotted in Figure 3 shows a steady decrease of the objective function during the optimization process.

Table 2: Optimal solutions using different approaches

Member number	Optimal cross-sectional area $\bar{A}_i$		
	Mixed model	Deterministic	Pure probabilistic model
1	42.91	31.37	39.23
2	0.10	0.10	0.10
3	29.32	21.48	26.81
4	21.01	15.46	19.23
5	0.10	0.10	0.10
6	0.10	0.10	0.10
7	3.38	2.83	3.21
8	30.81	22.56	28.18
9	29.94	21.86	27.36
10	0.10	0.10	0.10
Total weight	6638.0	4880.4	6076.9
b_{m}	3.00	<0	1.50

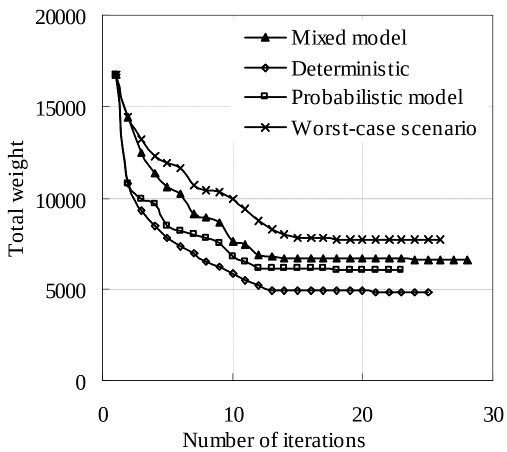

Figure 3: Iteration histories

5. Conclusions

This paper explores the reliability-based optimization design of non-deterministic structures with randomness and uncertain-but-bounded variations. The reliability-based optimization with constraints on such reliability indices is formulated as a nested optimization problem. By employing the performance measure approach, the original optimization problem is reformulated into an inherently more robust and numerically tractable one, in which the outer-loop aims to minimize the cost function while the inner-loop evaluates the performance value. The proposed optimization model proves to be capable of meeting the structural reliability requirements under mixed uncertainties.

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