Review for Exam 2
Optimization conditions

• Definitions of Global and local minima
  - We want to find a global but only afford to have local

• Unconstrained optimization problem
  - KT condition \( (f' = 0) \)
  - 2\textsuperscript{nd} order necessary condition \( (f'' \text{ PSD}) \)
  - Sufficient condition \( (f'' \text{ PD}) \)

• Condition for global minimum
  - Convex objective on convex constraint set
  - When the obj and constraint set become convex?

• Equality constrained problem
  - Introduce Lagrangian \( L(x, \lambda) = f(x) + \sum \lambda_i h_i(x) \)
  - KT condition
    \[
    \frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \lambda} = 0
    \]
Optimization conditions

• Inequality constrained problem
  - Introduce slack variables: \( L(x, \lambda, s) = f(x) + \sum \lambda_i (g_i + s_i^2) \)
  - KT condition
    \[
    \frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \lambda} = 0, \quad \frac{\partial L}{\partial s} = 0
    \]
    - Complementary slackness \( (\lambda_i g_i = 0) \)

• 2nd-order necessary condition
  - \( \nabla^2_L \) is P.S.D. for all feasible directions

• Sufficient condition
  - \( \nabla^2_L \) is P.D. for all feasible directions
Numerical Method for Optimization

- **Basic algorithm**
  - Move from one design to another until can’t reduce objective further
  - Need function values (objective & constraints) and their gradient
  - Need to find search direction and step size

\[ \Delta x^{(k)} = \alpha_k d^{(k)} \]

- **Unconstrained problem**
  - Descent condition: New objective function must be smaller than previous one
    \[ c^{(k)} \cdot d^{(k)} < 0 \]
  - Line search: find \( \alpha_k \) that minimize the objective function for given direction
    \[ \text{minimize } \phi(\alpha_k) = f(x^{(k)}) + \alpha_k d_k \]
  - Step size termination criterion:
    \[ c^{(k+1)} \cdot d^{(k)} = 0 \]
Numerical Method for Optimization

• Search direction
  - Search direction should reduce the objective function
  - Different algorithms are available for different ways of calculating the search direction

• Steepest descent method
  - The objective function can be reduced the most in the negative gradient direction
    \[ d^{(k)} = -c^{(k)} = -\nabla f^{(k)} \]
  - Although this method seems to reduce \( f(x) \) the most, its convergence is slow due to consecutive orthogonal search directions
    \[ c^{(k+1)} \perp c^{(k)} \]
  - This method converges slowly because the previous information is not used in finding the search direction
Numerical Method for Optimization

- Newton method
  - Very fast convergence when the initial design is close to the optimum design (quadratic convergence)
  - Need Hessian information $\Delta x^{(k)} = -H^{(k)-1} \cdot c^{(k)}$
  - If the Hessian in P.D., then new design will reduce $f(x)$
  - Difficulty in convergence when the Hessian changes its sign
  - Often line search is included (modified Newton method)

- Conjugate gradient method
  - Use previous gradient information $d^{(k)} = -c^{(k)} + \beta_k d^{(k-1)}$

- Quasi-Newton method
  - Calculating Hessian is expensive $\rightarrow$ Approximate Hessian or its inverse using gradient information
  - BFGS or DFP update
  - Maintain P.D. property of updated Hessian
Constrained Optimization

- Constrained optimization problem
  - Can convert to the unconstrained optimization problem
  - Can solve directly with constraints

- SUMT (Sequential Unconstrained Minimization Tech)
  - Penalize the objective function with violated constraints by multiplying with penalty parameter
  - Gradually increase the penalty parameter
  - When $r$ becomes too big, Hessian becomes ill-conditioned

- Lagrange multiplier method
  - Minimize Lagrangian with $x$ and $\lambda$
Constrained Optimization

• Direct method
  - Minimize the objective function with given feasible set
  - Can either follow interior or boundary of the feasible set
  - Epsilon-active strategy: for numerical purpose, consider a constraint active when it approaches zero

• Sequential linear programming (SLP)
  - Linearize the objective and constraints at the current design and solve for design change

• Quadratic programming subproblem (QP)
  - Quadratic objective with linear constraints for solving design change: convex problem and global optimum

• SLP and QP are used to calculate design change $\Delta x$, followed by line search for step size
Constrained Optimization

• Feasible direction method
  - Combine both feasible direction (satisfying constraints) and usable direction (reducing objective)

• Constrained quasi-Newton method (Sequential quadratic programming, SQP)
  - Solve the QP subproblem with approximate Hessian

\[
\minimize \quad f = c^T d + \frac{1}{2} d^T (\nabla_{xx} L) d \\
\text{s.t.} \quad N^T d = e \\
\quad \quad \quad A^T d \leq b
\]
  - Linear search for step size
Reliability Analysis and Design

• Study basic terminology of statistics (PDF, CDF, Normal…)

• Conditional probability
  \[ P(A | B) = \frac{P(A \cap B)}{P(B)} \]

• Transformation of RV (X \rightarrow Y)
  - For given statistical property of X, calculate property of Y
  - Linear transformation (Y = aX + b)
    \[ \mu_Y = a^T \mu_X + b \quad \sigma_Y^2 = a^T \Sigma_X a \]

• Nonlinear transformation (Y = g(X))
  - Linear approximation at mean: good when g is almost linear and uncertainty in X is small
  - Equivalent linearization: minimize expected value of square error
Transformation of Distribution

- **Monotonic function** $Y = g(X)$

- **CDF**
  \[
  F_Y(y) = \begin{cases} 
  F_X(g^{-1}(y)) & g \uparrow \\
  1 - F_X(g^{-1}(y)) & g \downarrow 
  \end{cases}
  \]

- **PDF**
  \[
  f_Y(y) = \begin{cases} 
  \frac{1}{g'(g^{-1}(y))} f_X(g^{-1}(y)) & g \uparrow \\
  -\frac{1}{g'(g^{-1}(y))} f_X(g^{-1}(y)) & g \downarrow 
  \end{cases}
  \]

- **General nonlinear function**
  - Need to find a region $g(x) \leq y$ and integrate $f_X$ on that region
  \[
  F_Y(y) = \int_{\{g(x) \leq y\}} f_X(x) \, dx
  \]
Reliability Analysis

• Limit state \( g(X) = 0 \); Failed state \( g(X) < 0 \)

• Probability of failure \( (P_F) \) and reliability index \( (\beta_{HL}) \)

\[
P_F = P[g(X) \leq 0] = \Phi(-\beta_{HL})
\]

• For general nonlinear limit state

\[
P_F = F_Y(0) = \int_{\{g(x) \leq 0\}} f_X(x) \, dx
\]

• Standard normal random variable with linear limit state

\[
g(U) = a_1 U_1 + a_2 U_2 + b
\]

\[
P_F = \Phi\left(-\frac{\mu_G}{\sigma_G}\right), \quad \frac{\mu_G}{\sigma_G} = \frac{b}{\sqrt{a_1^2 + a_2^3}}
\]
Approximate Reliability Analysis

- **First Order Reliability Analysis (FORM)**
  - Transform all input RVs (X) into SNRVs (U)
  - Find the closest point of \( g(U) = 0 \) from the origin
  - Approximate \( g(U) = 0 \) by tangent line at the closest point \( g_L(U) = 0 \)
  - Reliability index is the distance from origin to the closest point

- **Monte Carlo Simulation (MCS)**
  - General random samples of input RVs (\( N \))
  - Calculate the limit states samples using input RVs
  - Count the number of limit states less than zero (\( N_F \))

\[
P_{F,MCS} = \frac{N_F}{N}
\]
Reliability-based Design Optimization (RBDO)

- Reliability appears as a constraint in optimization
- Reliability index approach \( \beta_s = -\Phi^{-1}\{F_G(0)\} \geq \beta_t \)
  
  \[
  \begin{align*}
  \text{minimize} & \quad \|\mathbf{u}\| \\
  \text{subject to} & \quad G(\mathbf{u}) = 0 \\
  \end{align*}
  \]
  
  - Good for reliability analysis, but expensive and unstable when reliability is high or the limit state is highly nonlinear

- Performance measure approach \( G_p = F_G^{-1}\{\Phi(-\beta_t)\} \geq 0 \)
  
  \[
  \begin{align*}
  \text{maximize} & \quad G(\mathbf{u}) \\
  \text{subject to} & \quad \|\mathbf{u}\| = \beta_t \\
  \end{align*}
  \]
  
  - Not suitable for assessing reliability (\( \beta_t \) is fixed), but efficient and stable for design optimization

- Sensitivity of reliability can be calculated easily