CHAP 1
Preliminary Concepts and
Linear Finite Elements

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INTRODUCTION
Background

• Finite Element Method (FEM):
  - a powerful tool for solving partial differential equations and
    integro-differential equations
• Linear FEM:
  - methods of modeling and solution procedure are well established
• Nonlinear FEM:
  - different modeling and solution procedures based on the
    characteristics of the problems \( \rightarrow \) complicated
  - many textbooks in the nonlinear FEMs emphasize complicated
    theoretical parts or advanced topics
• This book:
  - to simply introduce the nonlinear finite element analysis procedure
    and to clearly explain the solution procedure
  - detailed theories, solution procedures, and implementation using
    MATLAB for only representative problems

Chapter Outline

2. Vector and Tensor Calculus
   - Preliminary to understand mathematical derivations in other
     chapters
3. Stress and Strain
   - Review of mechanics of materials and elasticity
4. Mechanics of Continuous Bodies
   - Energy principles for structural equilibrium (principle of minimum
     potential energy)
   - Principle of virtual work for more general non-potential problems
5. Finite Element Method
   - Discretization of continuum equations and approximation of
     solution using piecewise polynomials
   - Introduction to MATLAB program ELAST3D
1.2 VECTOR AND TENSOR CALCULUS

Vector and Tensor

- Vector: Collection of scalars
- Cartesian vector: Euclidean vector defined using Cartesian coordinates
  - 2D, 3D Cartesian vectors

\[ \mathbf{u} = \begin{\pmatrix} u_1 \\ u_2 \end{pmatrix}, \text{ or } \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \]

- Using basis vectors: \( \mathbf{e}_1 = \{1, 0, 0\}^T, \mathbf{e}_2 = \{0, 1, 0\}^T, \mathbf{e}_3 = \{0, 0, 1\}^T \)

\[ \mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3 \]
Index Notation and Summation Rule

- **Index notation**: Any vector or matrix can be expressed in terms of its indices
  
  \[ \mathbf{v} = [v_i] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \mathbf{A} = [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \]

- **Einstein summation convention**
  
  - In this case, \( k \) is a dummy variable (can be \( j \) or \( i \))
  
  \[ a_k b_k = a_j b_j \]

  - The same index cannot appear more than twice

- **Basis representation of a vector**
  - Let \( \mathbf{e}_k \) be the basis of vector space \( V \)
  
  - Then, any vector in \( V \) can be represented by
  
  \[ \mathbf{w} = \sum_{k=1}^{N} w_k \mathbf{e}_k = \mathbf{w}_k \mathbf{e}_k \]

Index Notation and Summation Rule cont.

- **Examples**
  
  - **Matrix multiplication**: \( \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \quad C_{ij} = A_{ik} B_{kj} \)
  
  - **Trace operator**: \( \text{tr}(\mathbf{A}) = A_{11} + A_{22} + A_{33} = A_{kk} \)
  
  - **Dot product**: \( \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = u_k v_k \)
  
  - **Cross product**: \( \mathbf{u} \times \mathbf{v} = u_j v_k (\mathbf{e}_j \times \mathbf{e}_k) = e_{ijk} u_j v_k \mathbf{e}_i \)

  \[ e_{ijk} = \begin{cases} 
  0 & \text{unless } i, j, k \text{ are distinct} \\
  +1 & \text{if } (i, j, k) \text{ is an even permutation} \\
  -1 & \text{if } (i, j, k) \text{ is an odd permutation} 
  \end{cases} \]

  - **Contraction**: double dot product

  \[ \mathbf{J} = \mathbf{A} : \mathbf{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} B_{ij} = A_{ij} B_{ij} \]
**Cartesian Vector**

- Cartesian Vectors
  \[
  \mathbf{u} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_3 \mathbf{e}_3 = u_i \mathbf{e}_i \\
  \mathbf{v} = v_j \mathbf{e}_j
  \]

- Dot product
  \[
  \mathbf{u} \cdot \mathbf{v} = (u_i \mathbf{e}_i) \cdot (v_j \mathbf{e}_j) = u_i v_j (\mathbf{e}_i \cdot \mathbf{e}_j) = u_i v_j \delta_{ij} = u_i v_i
  \]
  - Kronecker delta function
    \[
    \delta_{ij} = \begin{cases} 
    1 & \text{if } i = j \\
    0 & \text{if } i \neq j
    \end{cases}
    \]
    \[
    \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3
    \]
    - Equivalent to change index j to i, or vice versa

- How to obtain Cartesian components of a vector
  \[
  \mathbf{e}_i \cdot \mathbf{v} = e_i \cdot (v_j \mathbf{e}_j) = v_j \delta_{ij} = v_i
  \]
  \[
  \text{Projection}
  \]

- Magnitude of a vector (norm):
  \[
  \|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}
  \]

---

**Notation Used Here**

<table>
<thead>
<tr>
<th>Direct tensor notation</th>
<th>Tensor component notation</th>
<th>Matrix notation</th>
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</thead>
<tbody>
<tr>
<td>( \alpha = \mathbf{a} \cdot \mathbf{b} )</td>
<td>( \alpha = a_i b_i )</td>
<td>( \alpha = \mathbf{a}^T \mathbf{b} )</td>
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<tr>
<td>( \mathbf{A} = \mathbf{a} \otimes \mathbf{b} )</td>
<td>( A_{ij} = a_i b_j )</td>
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<td>( \mathbf{b} = \mathbf{A} \cdot \mathbf{a} )</td>
<td>( b_i = A_{ij} a_j )</td>
<td>( \mathbf{b} = \mathbf{Aa} )</td>
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<td>( \mathbf{b}^T = \mathbf{a}^T \mathbf{A} )</td>
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Tensor and Rank

• Tensor
  - A tensor is an extension of scalar, vector, and matrix (multidimensional array in a given basis)
  - A tensor is independent of any chosen frame of reference
  - Tensor field: a tensor-valued function associated with each point in geometric space

• Rank of Tensor
  - No. of indices required to write down the components of tensor
  - Scalar (rank 0), vector (rank 1), matrix (rank 2), etc
  - Every tensor can be expressed as a linear combination of rank 1 tensors

  - Rank 1 tensor $\mathbf{v}: v_i$
  - Rank 2 tensor $\mathbf{A}: A_{ij}$
  - Rank 4 tensor $\mathbf{C}: C_{ijkl}$

Tensor Operations

• Basic rules for tensors

  \[
  (TS)R = T(SR) \\
  T(S + R) = TS + TR \\
  \alpha(TS) = (\alpha T)S = T(\alpha S) \\
  1T = T1 = T
  \]

• Tensor (dyadic) product: increase rank

  \[
  A = u \otimes v = u v_j e_i \otimes e_j \\
  A_{ij} = u v_j \\
  A^T = A_{ji} e_i \otimes e_j \\
  (u \otimes v) \cdot w = u(v \cdot w) \\
  w \cdot (u \otimes v) = v(w \cdot u) \\
  (u \otimes v)(w \otimes x) = (v \cdot w)u \otimes x \\
  u \otimes v \neq v \otimes u
  \]

• Rank-4 tensor: $\mathbf{D} = D_{ijkl} e_i \otimes e_j \otimes e_k \otimes e_l$
Tensor Operations cont.

• Symmetric and skew tensors
  - Symmetric  \( S = S^T \)
  - Skew  \( W = -W^T \)
  - Every tensor can be uniquely decomposed by symmetric and skew tensors
    \[ T = S + W \]
    \[ S = \frac{1}{2}(T + T^T) \]
    \[ W = \frac{1}{2}(T - T^T) \]
  - Note: \( W \) has zero diagonal components and \( W_{ij} = -W_{ji} \)

• Properties - Let \( A \) be a symmetric tensor
  \[ A : W = 0 \]
  \[ A : T = A : S \]

Example

• Displacement gradient can be considered a tensor (rank 2)

\[ \nabla u = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \]

\[ \text{sym}(\nabla u) = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \quad \text{Strain tensor} \]

\[ \text{skew}(\nabla u) = \begin{bmatrix} 0 & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}) & 0 & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}) & 0 \end{bmatrix} \quad \text{Spin tensor} \]
Contraction and Trace

- Contraction of rank-2 tensors
  \[ \mathbf{a} : \mathbf{b} = a_{ij}b_{ij} = a_{11}b_{11} + a_{12}b_{12} + \ldots + a_{32}b_{32} + a_{33}b_{33} \]
  - contraction operator reduces four ranks from the sum of ranks of two tensors
- magnitude (or, norm) of a rank-2 tensor
  \[ |\mathbf{a}| = \sqrt{\mathbf{a} : \mathbf{a}} \]
- Constitutive relation between stress and strain
  \[ \sigma = \mathbf{D} : \mathbf{e}, \quad \sigma_{ij} = D_{ijkl}e_{kl} \]
- Trace: part of contraction
  \[ \text{tr}(\mathbf{A}) = A_{ii} = A_{11} + A_{22} + A_{33} \]
  - In tensor notation
  \[ \text{tr}(\mathbf{A}) = \mathbf{A} : 1 = 1 : \mathbf{A} \]

Orthogonal Tensor

- In two different coord.
  \[ \mathbf{u} = u_i \mathbf{e}_i = u_i^* \mathbf{e}_j^* \]
- Direction cosines
  \[ \mathbf{\beta} = [\beta_{ij}] = \mathbf{e}_i^* \otimes \mathbf{e}_j, \quad e_i^* = \beta e_j \]
- Change basis
  \[ \mathbf{u} = u_j \mathbf{e}_j = u_i^* \mathbf{e}_i^* = u_i^* \beta_{ij} \mathbf{e}_j \]
  \[ u_j = \beta_{ij} u_i^* \]
  \[ \mathbf{u} = \beta^T \mathbf{u}^* \]

We can also show
\[ \mathbf{e}_j = \beta_{ij} \mathbf{e}_i, \quad \mathbf{u}^* = \beta \mathbf{u} \]
\[ \mathbf{u} = \beta^T \mathbf{u}^* = \beta^T (\beta \mathbf{u}) = (\beta^T \beta)\mathbf{u} \]
\[ \beta^{-1} = \beta^T \]
\[ \beta^T \beta = \beta \beta^T = 1 \quad \det(\beta) = \pm 1 \]

Orthogonal tensor

Rank-2 tensor transformation
\[ \mathbf{T}^* = \beta \mathbf{T} \beta^T, \quad T_{ij}^* = \beta_{ik} T_{kl} \beta_{jl} \]
**Permutation**

- The permutation symbol has three indices, but it is not a tensor
  \[ e_{ijk} = \begin{cases} 
  1 & \text{if } ijk \text{ are an even permutation: 123, 231, 312} \\
  -1 & \text{if } ijk \text{ are an odd permutation: 132, 213, 321} \\
  0 & \text{otherwise} 
  \end{cases} \]

- The permutation is zero when any of two indices have the same value: \( e_{112} = e_{121} = e_{111} = 0 \)

- Identity
  \[ e_{ijk} e_{lmk} = \delta_{ij} \delta_{jm} - \delta_{im} \delta_{jl} \]

- Vector product
  \[ u \times v = e_{ijk} u_j v_k \]

**Dual Vector**

- For any skew tensor \( W \) and a vector \( u \)
  \[ u \cdot Wu = u \cdot W^T u = -u \cdot Wu = 0 \]
  - \( Wu \) and \( u \) are orthogonal

- Let \( W_{ij} = -e_{ijk} w_k \)
  \[ W = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix} \Rightarrow w = \begin{bmatrix} -W_{23} \\ W_{13} \\ -W_{12} \end{bmatrix} \]

- Then, \( W_{ij} u_j = -e_{ijk} w_k u_j = e_{ikj} w_k u_j \)
  \[ Wu = w \times u \]
  Dual vector of skew tensor \( W \)

  \[ w_i = -\frac{1}{2} e_{ijk} W_{jk} \]
Vector and Tensor Calculus

- **Gradient**
  \[ \nabla = \frac{\partial}{\partial \mathbf{X}} = e_i \frac{\partial}{\partial X_i} \]
  - Gradient is considered a vector
  - We will often use a simplified notation: \( v_{i,j} = \frac{\partial v_i}{\partial X_j} \)

- **Laplace operator**
  \[ \nabla^2 = \nabla \cdot \nabla = \left( e_i \frac{\partial}{\partial X_i} \right) \cdot \left( e_j \frac{\partial}{\partial X_j} \right) = \frac{\partial}{\partial X_j} \frac{\partial}{\partial X_j} \]

- **Gradient of a scalar field \( \phi(\mathbf{X}) \): vector**
  \[ \nabla \phi(\mathbf{X}) = e_i \frac{\partial \phi}{\partial X_i} \]
  \[ \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} + \frac{\partial^2}{\partial X_3^2} \]

- **Gradient of a Tensor Field (increase rank by 1)**
  \[ \nabla \phi = \phi \otimes \nabla = \phi_i e_i \otimes e_j \frac{\partial}{\partial X_j} = \frac{\partial \phi_i}{\partial X_j} e_i \otimes e_j \]

- **Divergence (decrease rank by 1)**
  \[ \nabla \cdot \phi = \left( e_i \frac{\partial}{\partial X_i} \right) \cdot \left( \phi_j e_j \right) = \frac{\partial \phi_i}{\partial X_i} \]
  - Ex) \( \nabla \cdot \mathbf{\sigma} = \sigma_{jk,j} e_k \)

- **Curl**
  \[ \nabla \times \mathbf{v} = e_i e_{ijk} v_{k,j} \]
Integral Theorems

- Divergence Theorem
\[ \iint_{\Omega} \nabla \cdot \mathbf{A} \, d\Omega = \int_{\Gamma} \mathbf{n} \cdot \mathbf{A} \, d\Gamma \]
\[ \text{n: unit outward normal vector} \]

- Gradient Theorem
\[ \iint_{\Omega} \nabla \mathbf{A} \, d\Omega = \int_{\Gamma} \mathbf{n} \otimes \mathbf{A} \, d\Gamma \]

- Stokes Theorem
\[ \int_{\Gamma} \mathbf{n} \cdot (\nabla \times \mathbf{v}) \, d\Gamma = \oint_{c} \mathbf{r} \cdot \mathbf{v} \, dc \]

- Reynolds Transport Theorem
\[ \frac{d}{dt} \iint_{\Omega} \mathbf{A} \, d\Omega = \iint_{\Omega} \frac{\partial \mathbf{A}}{\partial t} \, d\Omega + \int_{\Gamma} (\mathbf{n} \cdot \mathbf{v}) \mathbf{A} \, d\Gamma \]

Integration-by-Parts

- \( u(x) \) and \( v(x) \) are continuously differentiable functions
- 1D
\[ \int_{a}^{b} u(x)v'(x) \, dx = \left[ u(x)v(x) \right]_{a}^{b} - \int_{a}^{b} u'(x)v(x) \, dx \]

- 2D, 3D
\[ \int_{\Omega} \frac{\partial u}{\partial x_i} v \, d\Omega = \int_{\Gamma} u v n_i \, d\Gamma - \int_{\Omega} u \frac{\partial v}{\partial x_i} \, d\Omega \]

- For a vector field \( \mathbf{v}(x) \)
\[ \int_{\Omega} \nabla u \cdot \mathbf{v} \, d\Omega = \int_{\Gamma} u (\mathbf{v} \cdot \mathbf{n}) \, d\Gamma - \int_{\Omega} u \nabla \cdot \mathbf{v} \, d\Omega \]

- Green’s identity
\[ \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Gamma} u \nabla v \cdot \mathbf{n} \, d\Gamma - \int_{\Omega} u \nabla^2 v \, d\Omega \]
Example: Divergence Theorem

- S: unit sphere \((x^2 + y^2 + z^2 = 1)\), \(F = 2xi + y^2j + z^2k\)
- Integrate \(\int_S F \cdot n \, dS\)

\[
\int_S F \cdot n \, dS = \iiint_{\Omega} \nabla \cdot F \, d\Omega
\]

\[
= 2\iiint_{\Omega} (1 + y + z) \, d\Omega
\]

\[
= 2\iiint_{\Omega} d\Omega + 2\iiint_{\Omega} y \, d\Omega + 2\iiint_{\Omega} z \, d\Omega
\]

\[
= 2\iiint_{\Omega} d\Omega
\]

\[
= \frac{8\pi}{3}
\]

1.3

STRESS AND STRAIN
Surface Traction (Stress)

- **Surface traction (Stress)**
  - The entire body is in equilibrium with external forces \((f_1 \sim f_6)\)
  - The imaginary cut body is in equilibrium due to external forces \((f_1, f_2, f_3)\) and internal forces
  - Internal force acting at a point \(P\) on a plane whose unit normal is \(n\):
    \[
    t^{(n)} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}
    \]
  - The surface traction depends on the unit normal direction \(n\).
  - Surface traction will change as \(n\) changes.
    - unit = force per unit area (pressure)
    \[
    t^{(n)} = t_1 e_1 + t_2 e_2 + t_3 e_3
    \]

Cartesian Stress Components

- **Surface traction changes according to the direction of the surface.**
- Impossible to store stress information for all directions.
- Let’s store surface traction parallel to the three coordinate directions.
- Surface traction in other directions can be calculated from them.
- Consider the \(x\)-face of an infinitesimal cube
  \[
  t^{(x)} = t_1^{(x)} e_1 + t_2^{(x)} e_2 + t_3^{(x)} e_3
  \]
  \[
  t^{(x)} = \sigma_{11} e_1 + \sigma_{12} e_2 + \sigma_{13} e_3
  \]
  - Normal stress
  - Shear stress
Stress Tensor

- First index is the face and the second index is its direction
- When two indices are the same, normal stress, otherwise shear stress.
- Continuation for other surfaces.
- Total nine components
- Same stress components are defined for the negative planes.

• Rank-2 Stress Tensor

\[ \sigma = \sigma_{ij} e_i \otimes e_j \]

• Sign convention

\[
\text{sgn}(\sigma_{11}) = \text{sgn}(n) \times \text{sgn}(\Delta F_x) \\
\text{sgn}(\sigma_{12}) = \text{sgn}(n) \times \text{sgn}(\Delta F_y)
\]

Symmetry of Stress Tensor

- Stress tensor should be symmetric
  9 components \rightarrow 6 components
- Equilibrium of the angular moment
  \[
  \sum M = \Delta l (\sigma_{12} - \sigma_{21}) = 0 \\
  \Rightarrow \sigma_{12} = \sigma_{21}
  \]
- Similarly for all three directions:
  \[
  \sigma_{12} = \sigma_{21}, \quad \sigma_{23} = \sigma_{32}, \quad \sigma_{13} = \sigma_{31}
  \]
- Let's use vector notation:

\[
\{\sigma\} = \begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{23} \\
\sigma_{13}
\end{bmatrix} \\
\begin{bmatrix}
\sigma_{ij}
\end{bmatrix} = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
\end{bmatrix}
\]
Stress in Arbitrary Plane

- If Cartesian stress components are known, it is possible to determine the surface traction acting on any plane.
- Consider a plane whose normal is $\mathbf{n}$.
- Surface area ($\Delta ABC = A$)

$$\Delta PAB = A_n_3; \; \Delta PBC = A_n_1; \; \Delta PAC = A_n_2$$

- The surface traction

$$\mathbf{t}^{(n)} = t_1^{(n)} \mathbf{e}_1 + t_2^{(n)} \mathbf{e}_2 + t_3^{(n)} \mathbf{e}_3$$

- Force balance

$$\sum F_1 = t_1^{(n)} A - \sigma_{11} A_n_1 - \sigma_{21} A_n_2 - \sigma_{31} A_n_3 = 0$$

$$t_1^{(n)} = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$

Cauchy's Lemma

- All three-directions

$$t_1^{(n)} = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$
$$t_2^{(n)} = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$$
$$t_3^{(n)} = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$

- Tensor notation

$$\mathbf{t}^{(n)} = \mathbf{n} \cdot \sigma \; \Rightarrow \; \mathbf{t}^{(n)} = \sigma \cdot \mathbf{n}$$

- stress tensor; completely characterize the state of stress at a point

- Cauchy's Lemma

- the surface tractions acting on opposite sides of the same surface are equal in magnitude and opposite in direction

$$\mathbf{t}^{(n)} = -\mathbf{t}^{(-n)}$$
Projected Stresses

- Normal stress $\sigma(n) = t^n \cdot n = n \cdot \sigma \cdot n = \sigma_{ij} n_i n_j$
- Shear stress $\tau(n) = \sqrt{\|t^n\|^2 - \sigma^2(n)}$, $\tau(n) = t^n - n \sigma(n)$
- Principal stresses $t^n \parallel n \Rightarrow \sigma_1, \sigma_2, \sigma_3$
- Mean stress (hydrostatic pressure)
  $$p = \sigma_m = \frac{1}{3} \text{tr}(\sigma) = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$
- Stress deviator
  $$s = \sigma - \sigma_m 1 = I_{\text{dev}} : \sigma$$
  $$s = \begin{bmatrix} \sigma_{11} - \sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_m & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_m \end{bmatrix}$$

$$I_{ijkl} = \frac{1}{3}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$
Rank-4 identity tensor

$$I_{\text{dev}} = I - \frac{1}{3} 1 \otimes 1$$
Rank-4 deviatoric identity tensor

$$I_{\text{dev}} : 1 = 0, \quad I_{\text{dev}} : s = s$$

Principal Stresses

- Normal & shear stress change as $n$ changes
  - Is there a plane on which the normal (or shear)stress becomes the maximum?
- There are at least three mutually perpendicular planes on which the normal stress attains an extremum
  - Shear stresses are zero on these planes $\Rightarrow$ Principal directions
  - Traction $t^{(n)}$ is parallel to surface normal $n$
  $$t^{(n)} = \sigma_n n \quad \Rightarrow \quad \sigma \cdot n = \sigma_n n$$
- Eigenvalue problem
  $$[\sigma - \sigma_n 1] \cdot n = 0$$
  $$\begin{bmatrix} \sigma_{11} - \sigma_n & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_n & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_n \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
Eigenvalue Problem for Principal Stresses

- The eigenvalue problem has non-trivial solution if and only if the determinant is zero:

\[
\begin{vmatrix}
\sigma_{11} - \sigma_n & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} - \sigma_n & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_n
\end{vmatrix} = 0
\]

- The above equation becomes a cubic equation:

\[
\sigma_n^3 - I_1 \sigma_n^2 + I_2 \sigma_n - I_3 = 0
\]

\[
I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}
\]

\[
I_2 = \sigma_{11} \sigma_{22} + \sigma_{22} \sigma_{33} + \sigma_{33} \sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2
\]

\[
I_3 = |\sigma| = \sigma_{11} \sigma_{22} \sigma_{33} + 2 \sigma_{12} \sigma_{23} \sigma_{13} - \sigma_{11} \sigma_{23}^2 - \sigma_{22} \sigma_{13}^2 - \sigma_{33} \sigma_{12}^2
\]

- Three roots are principal stresses

\[
\sigma_1 \geq \sigma_2 \geq \sigma_3
\]

Principal Directions

- Stress Invariants: \(I_1, I_2, I_3\)
  - independent of the coordinate system
- Principal directions
  - Substitute each principal stress to the eigenvalue problem to get \(n\)
  - Since the determinant is zero, an infinite number of solutions exist
  - Among them, choose the one with a unit magnitude

\[
\|n^i\|^2 = (n_1^i)^2 + (n_2^i)^2 + (n_3^i)^2 = 1, \quad i = 1, 2, 3
\]

- Principal directions are mutually perpendicular

\[
n^i \cdot n^j = 0, \quad i \neq j
\]
Principal Directions

- There are three cases for principal directions:
  1. $\sigma_1, \sigma_2, \text{ and } \sigma_3$ are distinct $\Rightarrow$ principal directions are three unique mutually orthogonal unit vectors.
  2. $\sigma_1 = \sigma_2$ and $\sigma_3$ are distinct $\Rightarrow n^3$ is a unique principal direction, and any two orthogonal directions on the plane that is perpendicular to $n^3$ are principal directions.
  3. $\sigma_1 = \sigma_2 = \sigma_3 \Rightarrow$ any three orthogonal directions are principal directions. This state of stress corresponds to a hydrostatic pressure.

Strains (Simple Version)

- Strain is defined as the elongation per unit length

- Tensile (normal) strains in $x_1$- and $x_2$-directions

  \[ \varepsilon_{11} = \lim_{\Delta x_1 \to 0} \frac{\Delta u_1}{\Delta x_1} = \frac{\partial u_1}{\partial x_1} \]

  \[ \varepsilon_{22} = \lim_{\Delta x_2 \to 0} \frac{\Delta u_2}{\Delta x_2} = \frac{\partial u_2}{\partial x_2} \]

- Strain is a dimensionless quantity. Positive for elongation and negative for compression.
Shear Strain

- Shear strain is the tangent of the change in angle between two originally perpendicular axes

\[ \theta_1 \sim \tan \theta_1 = \frac{\Delta u_2}{\Delta x_1} \]

\[ \theta_2 \sim \tan \theta_2 = \frac{\Delta u_1}{\Delta x_2} \]

- Shear strain (change of angle)

\[ \gamma_{12} = \theta_1 + \theta_2 = \lim_{\Delta x_1 \to 0} \frac{\Delta u_2}{\Delta x_1} + \lim_{\Delta x_2 \to 0} \frac{\Delta u_1}{\Delta x_2} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \]

\[ \varepsilon_{12} = \frac{1}{2} \gamma_{12} = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) \]

- Positive when the angle between two positive (or two negative) faces is reduced and negative when the angle is increased.

- Valid for small deformation

Strains (Rigorous Version)

- Strain: a measure of deformation
  - Normal strain: change in length of a line segment
  - Shear strain: change in angle between two perpendicular line segments

- Displacement of \( P = (u_1, u_2, u_3) \)

- Displacement of \( Q \) & \( R \)

\[ u_1^Q = u_1 + \frac{\partial u_1}{\partial x_1} \Delta x_1 \quad u_1^R = u_1 + \frac{\partial u_1}{\partial x_2} \Delta x_2 \]

\[ u_2^Q = u_2 + \frac{\partial u_2}{\partial x_1} \Delta x_1 \quad u_2^R = u_2 + \frac{\partial u_2}{\partial x_2} \Delta x_2 \]

\[ u_3^Q = u_3 + \frac{\partial u_3}{\partial x_1} \Delta x_1 \quad u_3^R = u_3 + \frac{\partial u_3}{\partial x_2} \Delta x_2 \]
**Displacement Field**

- Coordinates of P, Q, and R before and after deformation

  \[ P : (x_1, x_2, x_3) \]
  \[ Q : (x_1 + \Delta x_1, x_2, x_3) \]
  \[ R : (x_1, x_1 + \Delta x_2, x_3) \]

  \[ P' : (x_1 + u^P, x_2, x_3 + u^P) = (x_1 + u_1, x_2 + u_2, x_3 + u_3) \]

  \[ Q' : (x_1 + \Delta x_1 + u^Q, x_2 + u^Q_2, x_3 + u^Q_3) \]

  \[ R' : (x_1 + u^R, x_2 + \Delta x_2 + u^R_2, x_3 + u^R_3) \]

- Length of the line segment \( P'Q' \)

  \[ P'Q' = \sqrt{\left( x'_1 - x'^Q_1 \right)^2 + \left( x'_2 - x'^Q_2 \right)^2 + \left( x'_3 - x'^Q_3 \right)^2} \]

**Deformation Field**

- Length of the line segment \( P'Q' \)

  \[ P'Q' = \Delta x_1 \sqrt{\left( 1 + \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \left( \frac{\partial u_3}{\partial x_1} \right)^2} \]

  \[ = \Delta x_1 \left( 1 + 2 \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}^2 + \frac{\partial u_2}{\partial x_1}^2 + \frac{\partial u_3}{\partial x_1}^2 \right)^{1/2} \]

  \[ \approx \Delta x_1 \left( 1 + \frac{\partial u_1}{\partial x_1} + \frac{1}{2} \left( \frac{\partial u_1}{\partial x_1} \right)^2 \right) \approx \Delta x \left( 1 + \frac{\partial u_1}{\partial x_1} \right) \]

  Linear

  Nonlinear \( \Rightarrow \) Ignore H.O.T. when displacement gradients are small

- Linear normal strain

  \[ \varepsilon_{11} = \frac{P'Q' - PQ}{PQ} = \frac{\partial u_1}{\partial x_1} \]

  \[ \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \]
Deformation Field

• Shear strain $\gamma_{xy}$
  - change in angle between two lines originally parallel to x- and y-axes
  \[ \theta_1 = \frac{x_2^Q - x_2^Q}{\Delta x_1} = \frac{\partial u_2}{\partial x_1} \]
  \[ \theta_2 = \frac{x_1^R - x_1^R}{\Delta x_2} = \frac{\partial u_1}{\partial x_2} \]
  \[ \gamma_{12} = \theta_1 + \theta_2 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \]
  \[ \gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \]
  \[ \gamma_{13} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \]

Engineering shear strain

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]
\[ \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]
\[ \varepsilon = \text{sym}(\nabla u) \]

Strain Tensor

• Strain Tensor
  \[ \varepsilon = \varepsilon_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \]

• Cartesian Components
  \[ [\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \]

• Vector notation
  \[ \{\varepsilon\} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2 \varepsilon_{12} \\ 2 \varepsilon_{23} \\ 2 \varepsilon_{13} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{bmatrix} \begin{bmatrix} \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix} \]
Volumetric and Deviatoric Strain

- Volumetric strain (from small strain assumption)
  \[ \varepsilon_V = \frac{V - V_0}{V_0} = (1 + \varepsilon_{11})(1 + \varepsilon_{22})(1 + \varepsilon_{33}) - 1 \approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \]
  \[ \varepsilon_V = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{kk} \]

- Deviatoric strain
  \[ \mathbf{e} = \varepsilon - \frac{1}{3} \varepsilon_V \mathbf{1} \]
  \[ \varepsilon_{ij} = \varepsilon_{ij} - \frac{1}{3} \varepsilon_V \delta_{ij} \]
  \[ \mathbf{e} = \mathbf{I}_{dev} \cdot \varepsilon \]

Exercise: Write \( \mathbf{I}_{dev} \) in matrix-vector notation

Stress-Strain Relationship

- Applied Load \( \rightarrow \) shape change (strain) \( \rightarrow \) stress
- There must be a relation between stress and strain
- Linear Elasticity: Simplest and most commonly used
**Generalized Hooke's Law**

- **Linear elastic material** \( \sigma = \mathbf{D} : \mathbf{\varepsilon} \), \( \sigma_{ij} = D_{ijkl}\varepsilon_{kl} \)
  
  - In general, \( D_{ijkl} \) has 81 components
  
  - Due to symmetry in \( \sigma_{ij} \), \( D_{ijkl} = D_{jikl} \)
  
  - Due to symmetry in \( \varepsilon_{kl} \), \( D_{ijkl} = D_{ijlk} \)
  
  - From definition of strain energy, \( D_{ijkl} = D_{klij} \)

- **Isotropic material** (no directional dependence)
  
  - Most general 4-th order isotropic tensor
  
  \[
  D_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})
  \]

  - Have only two independent coefficients
  
  (Lame’s constants: \( \lambda \) and \( \mu \))

\[
\mathbf{D} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}
\]

**Generalized Hooke’s Law cont.**

- **Stress-strain relation**
  
  \[
  \sigma_{ij} = D_{ijkl}\varepsilon_{kl} = [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\varepsilon_{kl} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}
  \]

  - Volumetric strain: \( \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_\varepsilon \)

  - Off-diagonal part: \( \sigma_{12} = 2\mu\varepsilon_{12} = \mu\gamma_{12} \) \( \mu \) is the shear modulus

  - Bulk modulus \( K \): relation b/w volumetric stress & strain

  \[
  I_1 = 3\sigma_m = \sigma_{jj} = \lambda\varepsilon_{kk}\delta_{jj} + 2\mu\varepsilon_{jj} = (3\lambda + 2\mu)\varepsilon_{kk}
  \]

  \[
  p = \sigma_m = \left(\lambda + \frac{2}{3}\mu\right)\varepsilon_{kk} = K\varepsilon_\varepsilon
  \]

  - Substitute \( \lambda = K - \frac{2}{3}\mu \) so that we can separate volumetric part

- **Total deform.** = volumetric + deviatoric deform.
Generalized Hooke’s Law cont.

• Stress-strain relation cont.

\[ \sigma_{ij} = (K - \frac{2}{3} \mu) \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \]
\[ = K \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} - \frac{2}{3} \mu \varepsilon_{kk} \delta_{ij} \]
\[ = K \delta_{ij} \delta_{kl} \varepsilon_{kl} + 2\mu [\delta_{ik} \delta_{jl} - \frac{1}{3} \delta_{ij} \delta_{kl}] \varepsilon_{kl} \]
\[ = \left[ K \delta_{ij} \delta_{kl} + 2\mu (I_{dev})_{jkl} \right] \varepsilon_{kl} \]

\[ \Rightarrow \sigma = \left[ K(1) \otimes 1 + 2\mu I_{dev} \right] : \varepsilon \]

\[ \Rightarrow \sigma = K \varepsilon_v 1 + 2\mu e \]
\[ \sigma = \sigma_m 1 + s \]

Important for plasticity; plastic deformation only occurs in deviatoric part
volumetric part is always elastic

Generalized Hooke’s Law cont.

• Vector notation

- The tensor notation is not convenient for computer implementation
- Thus, we use Voigt notation
  2nd-order tensor ⇒ vector
  4th-order tensor ⇒ matrix
- Strain (6×1 vector), Stress (6×1 vector), and \( C \) (6×6 matrix)

\[ \varepsilon = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix} = \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{1,2} + u_{2,1} \\ u_{2,3} + u_{3,2} \\ u_{1,3} + u_{3,1} \end{bmatrix} \]
\[ \sigma = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} + \sigma_{21} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = D\varepsilon \]

You don’t need 2 here
3D Solid Element cont.

- Elasticity matrix
  \[ \mathbf{D} = K I \otimes \mathbf{1} + 2 \mu \mathbf{I}_{\text{dev}} \]
  \[ \mathbf{I}_{\text{dev}} = \begin{bmatrix}
  \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\
  -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\
  -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\
  0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
  0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
  0 & 0 & 0 & 0 & 0 & \mu 
\end{bmatrix} \]

- Relation b/w Lame's constants and Young's modulus and Poisson's ratio
  \[ \nu = \frac{\lambda}{2(\lambda + \mu)}, \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \lambda = \frac{E V}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)} \]

Textbook has a definition of \( \mathbf{D} \) in terms of \( E \) and \( \nu \)

Plane Stress

- Thin plate-like components parallel to the xy-plane
- The plate is subjected to forces in its in-plane only
  - \( \sigma_{13} = \sigma_{23} = \sigma_{33} = 0 \)

  \[ \{\sigma\} = \begin{bmatrix}
  \sigma_{11} \\
  \sigma_{22} \\
  \sigma_{12}
\end{bmatrix} = \frac{E}{1 - \nu^2} \begin{bmatrix}
  1 & \nu & 0 \\
  \nu & 1 & 0 \\
  0 & 0 & \frac{1}{2}(1 - \nu)
\end{bmatrix} \begin{bmatrix}
  \varepsilon_{11} \\
  \varepsilon_{22} \\
  \gamma_{12}
\end{bmatrix} \]

- \( \varepsilon_{13} = \varepsilon_{23} = 0 \), but \( \varepsilon_{33} \neq 0 \)
- \( \varepsilon_{33} \) can be calculated from the condition of \( \sigma_{33} = 0 \):
  \[ \varepsilon_{33} = -\frac{\nu}{1 - \nu}(\varepsilon_{11} + \varepsilon_{22}) \]
Plane Strain

- Strains with a z subscript are all zero: \( \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0 \)
- Deformation in the z-direction is constrained, (i.e., \( u_3 = 0 \))
- can be used if the structure is infinitely long in the z-direction

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix} = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu & \nu & 0 \\
\nu & 1 - \nu & 0 \\
0 & 0 & \frac{1}{2} - \nu
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\gamma_{12}
\end{bmatrix}
\]

- \( \sigma_{13} = \sigma_{23} = 0 \), but \( \sigma_{33} \neq 0 \)
- \( \sigma_{33} \) can be calculated from the condition of \( \varepsilon_{33} = 0 \):

\[
\sigma_{33} = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} (\varepsilon_{11} + \varepsilon_{22})
\]
Governing Equations for Equilibrium

- Governing differential equations for structural equilibrium
  - Three laws of mechanics: conservation of mass, conservation of linear momentum and conservation of angular momentum
- Boundary-valued problem: satisfied at every point in \( \Omega \)
  - Governing D.E. + Boundary conditions
  - Solutions: \( C^2 \)-continuous for truss & solid, \( C^4 \)-continuous for beam
  - Unnecessary requirements for higher-order continuity
- Energy-based method
  - For conservative system, structural equilibrium when the potential energy has its minimum: Principle of minimum potential energy
  - If the solution of BVP exists, then that solution is the minimizing solution of the potential energy
  - When no solution exists in BVP, PMPE may have a natural solution
- Principle of virtual work
  - Equilibrium of the work done by both internal and external forces with small arbitrary virtual displacements

Balance of Linear Momentum

- Balance of linear momentum
  \[
  \int \int_\Omega \mathbf{f}^b \, d\Omega + \int_\Gamma \mathbf{t}^n \, d\Gamma = \int \int_\Omega \rho \mathbf{a} \, d\Omega \quad \text{for static problem}
  \]
  - \( \mathbf{f}^b \): body force
  - \( \mathbf{t}^n \): surface traction
- Stress tensor (rank 2):
  \[
  \sigma = \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \quad \begin{bmatrix} \sigma_{ij} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}
  \]
- Surface traction
  \[ \mathbf{t}^n = \mathbf{n} \cdot \sigma \]
- Cauchy’s Lemma
  \[
  \mathbf{t}^n = -\mathbf{t}^{-n} \\
  \mathbf{t}^n = \mathbf{n} \cdot \sigma \quad \mathbf{t}^{-n} = -\mathbf{n} \cdot \sigma
  \]
Balance of Linear Momentum cont

- Balance of linear momentum
  \[ \iint_{\Omega} (f^b - \rho a) \, d\Omega = -\oint_{\Gamma} \mathbf{n} \cdot \mathbf{\sigma} \, d\Gamma = -\iint_{\Omega} \nabla \cdot \mathbf{\sigma} \, d\Omega \]
  \[ \iint_{\Omega} [\nabla \cdot \mathbf{\sigma} + (f^b - \rho a)] \, d\Omega = 0 \]
  \[ \implies \nabla \cdot \mathbf{\sigma} + (f^b - \rho a) = 0 \]
  - For a static problem
    \[ \nabla \cdot \mathbf{\sigma} + f^b = 0 \quad \sigma_{ij,i} + f^b_j = 0 \]

- Balance of angular momentum
  \[ \iint_{\Omega} \mathbf{x} \times f^b \, d\Omega + \oint_{\Gamma} \mathbf{x} \times t^b \, d\Gamma = \iint_{\Omega} \rho \mathbf{x} \times a \, d\Omega \]
  \[ \implies \mathbf{\sigma} = \mathbf{\sigma}^T \quad \sigma_{ij} = \sigma_{ji} \]

Boundary-Valued Problem

- We want to determine the state of a body in equilibrium
- The equilibrium state (solution) of the body must satisfy
  - local momentum balance equation
  - boundary conditions

  **Strong form** of BVP

  - Given body force \( f^b \), and traction \( t^b \) on the boundary, find \( u \) such that
    \[ \nabla \cdot \mathbf{\sigma} + f^b = 0 \] (1)
    and
    \[ u = 0 \quad \text{on } \Gamma^h \quad \text{essential BC} \] (2)
    \[ t = n \cdot \mathbf{\sigma} \quad \text{on } \Gamma^s \quad \text{natural BC} \] (3)

- Solution space
  \[ D_A = \left\{ u \in [C^2(\Omega)]^3 \mid u = 0 \quad \text{on } x \in \Gamma^h, \quad \sigma \cdot n = t \quad \text{on } x \in \Gamma^s \right\} \]
Boundary-Valued Problem cont.

- **How to solve BVP**
  - To solve the strong form, we want to construct trial solutions that automatically satisfy a part of BVP and find the solution that satisfy remaining conditions.
  - **Statically admissible stress field**: satisfy (1) and (3)
  - **Kinematically admissible displacement field**: satisfy (2) and have piecewise continuous first partial derivative
  - Admissible stress field is difficult to construct. Thus, admissible displacement field is used often

Principle of Minimum Potential Energy (PMPE)

- Deformable bodies generate internal forces by deformation against externally applied forces
- **Equilibrium**: balance between internal and external forces
- For elastic materials, the concept of force equilibrium can be extended to energy balance
- **Strain energy**: stored energy due to deformation (corresponding to internal force)

\[
U(u) = \frac{1}{2} \int \int_{\Omega} \sigma(u) : \varepsilon(u) \, d\Omega \\
\sigma(u) = D : \varepsilon(u)
\]

Linear elastic material

- For elastic material, \(U(u)\) is only a function of total displacement \(u\) (independent of path)
• Work done by applied loads (conservative loads)

\[ W(u) = \iint_{\Omega} \mathbf{u} \cdot \mathbf{f}^b \, d\Omega + \int_{\Gamma^s} \mathbf{u} \cdot \mathbf{t} \, d\Gamma. \]

• \( U(u) \) is a quadratic function of \( u \), while \( W(u) \) is a linear function of \( u \).

• Potential energy

\[ \Pi(u) = U(u) - W(u) = \frac{1}{2} \iint_{\Omega} \mathbf{\sigma}(u) : \mathbf{\varepsilon}(u) \, d\Omega - \iint_{\Omega} \mathbf{u} \cdot \mathbf{f}^b \, d\Omega - \int_{\Gamma^s} \mathbf{u} \cdot \mathbf{t} \, d\Gamma. \]

• PMPE: for all displacements that satisfy the boundary conditions, known as kinematically admissible displacements, those which satisfy the boundary-valued problem make the total potential energy stationary on \( D_A \).

<table>
<thead>
<tr>
<th>PMPE cont.</th>
</tr>
</thead>
<tbody>
<tr>
<td>PMPE: for all displacements that satisfy the boundary conditions, known as kinematically admissible displacements, those which satisfy the boundary-valued problem make the total potential energy stationary on ( D_A ).</td>
</tr>
<tr>
<td>But, the potential energy is well defined in the space of kinematically admissible displacements</td>
</tr>
<tr>
<td>( \mathbb{Z} = \left{ \mathbf{u} \in [H^1(\Omega)]^3 \mid \mathbf{u} = \mathbf{0} \text{ on } \mathbf{x} \in \Gamma^h \right} , )</td>
</tr>
<tr>
<td>( H^1: \text{first-order derivatives are integrable} )</td>
</tr>
<tr>
<td>No need to satisfy traction BC (it is a part of potential)</td>
</tr>
<tr>
<td>Less requirement on continuity</td>
</tr>
<tr>
<td>The solution is called a generalized (natural) solution</td>
</tr>
</tbody>
</table>
Example – Uniaxial Bar

- Strong form
  \[ EAu'' = 0 \quad x \in [0, L] \]
  \[ u = 0 \quad x = 0 \]
  \[ EAU'(L) = F \quad x = L \]

- Integrate twice: \( EAu(x) = c_1 x + c_2 \)
- Apply two BCs: \( u(x) = \frac{Fx}{EA} \) Solution of BVP
- PMPE with assumed solution \( u(x) = c_1 x + c_2 \)
- To satisfy KAD space, \( u(0) = 0 \Rightarrow u(x) = c_1 x \)
- Potential energy: \( U = \frac{1}{2} \int_0^L EA(u')^2 \, dx = EALc_1^2 \)
  \[ W = Fu(L) = FLC_1 \]
  \[ \frac{d\Pi}{dc_1} = \frac{d}{dc_1} (U - W) = EALc_1 - FL = 0 \quad c_1 = \frac{F}{EA} \quad \Rightarrow \quad u(x) = \frac{Fx}{EA} \]

Virtual Displacement

- Virtual displacement is not experienced but only assumed to exist so that various possible equilibrium positions may be compared to determine the correct one
- Let mass \( m \) and springs are in equilibrium at the current position
- Then, a small arbitrary perturbation, \( \delta r \), can be assumed
  - Since \( \delta r \) is so small, the member forces are assumed unchanged
- The work done by virtual displacement is
  \[ \delta W = F_1 \cdot \delta r + F_2 \cdot \delta r + F_3 \cdot \delta r + F_4 \cdot \delta r = (F_1 + F_2 + F_3 + F_4) \cdot \delta r \]
- If the current position is in force equilibrium, \( \delta W = 0 \)
Virtual Displacement Field

- Virtual displacement (Space \( \mathcal{Z} \))
  - Small arbitrary perturbation (variation) of real displacement
    \[
    \delta u = \lim_{\tau \to 0} \left[ (u + \tau \eta) - u \right] = \frac{d}{d\tau} (u + \tau \eta) \bigg|_{\tau = 0} = \eta \equiv \bar{u}.
    \]
  - Let \( \bar{u} \) be the virtual displacement, then \( u + \bar{u} \) must be kinematically admissible, too
  - Then, \( \bar{u} \) must satisfy homogeneous displacement BC
    \[
    u \mapsto u + \tau \bar{u} \in \mathcal{V} \quad \Rightarrow \quad \bar{u} \in \mathcal{Z}
    \]
    \[
    \mathcal{Z} = \left\{ \bar{u} \mid \bar{u} \in [H^1(\Omega)^3], \; \bar{u} \big|_{\Gamma^h} = 0 \right\}
    \]
  - Space \( \mathcal{Z} \) only includes homogeneous essential BCs
    - In the literature, \( \delta u \) is often used instead of \( \bar{u} \)

- Property of variation
  \[
  \delta \left( \frac{du}{dx} \right) = \frac{d(\delta u)}{dx}
  \]

PMPE As a Variation

- Necessary condition for minimum PE
  - Stationary condition \( \Rightarrow \) first variation = 0
    \[
    \delta \Pi(u; \bar{u}) = \lim_{\tau \to 0} \frac{1}{\tau} \left[ \Pi(u + \tau \bar{u}) - \Pi(u) \right] = \frac{d}{d\tau} \Pi(u + \tau \bar{u}) \bigg|_{\tau = 0} = 0
    \]
    for all \( \bar{u} \in \mathcal{Z} \)

- Variation of strain energy
  \[
  \delta \left( \frac{\partial u}{\partial x} \right) = \frac{d}{d\tau} \left( \frac{\partial u + \tau \bar{u}}{\partial x} \right) \bigg|_{\tau = 0} = \frac{\partial \bar{u}}{\partial x}
  \]
  \[
  \delta \varepsilon(u) = \varepsilon(\bar{u}) = \bar{\varepsilon}
  \]
  \[
  \delta \sigma = D : \bar{\varepsilon}
  \]
  \[
  \delta U(u; \bar{u}) = \frac{1}{2} \iint_{\Omega} \left[ \varepsilon(\bar{u}) : D : \varepsilon(u) + \varepsilon(u) : D : \varepsilon(\bar{u}) \right] d\Omega
  \]
  \[
  = \iint_{\Omega} \varepsilon(\bar{u}) : D : \varepsilon(u) d\Omega
  \]
  \[
  = a(u, \bar{u}) \quad \text{Energy bilinear form}
  \]
**PMPE As a Variation cont.**

- Variation of work done by applied loads

\[
\delta W(u; \bar{u}) = \int_\Omega \bar{u} \cdot f \, d\Omega + \int_{\Gamma^e} \bar{u} \cdot t \, d\Gamma = \ell(\bar{u}) \quad \text{Load linear form}
\]

- Thus, PMPE becomes

\[
\delta \Pi(u; \bar{u}) = \delta U(u; \bar{u}) - \delta W(u; \bar{u}) = 0
\]

- **How can we satisfy “for all \( \bar{u} \in \mathbb{Z} \)” requirement?**

**Example – Uniaxial Bar**

- Assumed displacement \( u(x) = cx \rightarrow \)
  - virtual displacement is in the same space with \( u(x) \): \( \bar{u}(x) = \bar{c}x \)

- Variation of strain energy

\[
\delta U = \frac{d}{d\tau} \left[ \frac{1}{2} \int_0^L EA \left[ (u + \tau \bar{u})' \right]^2 \, dx \right] \bigg|_{\tau=0} = \frac{1}{2} \int_0^L 2EA(u + \tau \bar{u})' \bar{u}' \, dx \bigg|_{\tau=0} = \int_0^L EA u' \bar{u}' \, dx = EALc\bar{c}
\]

- Variation of applied load

\[
\delta W = \frac{d}{d\tau} \left[ F \left[ u(L) + \tau \bar{u}(L) \right] \right] \bigg|_{\tau=0} = F\bar{u}(L) = FL\bar{c}
\]

- **PMPE**

\[
\delta \Pi = \delta U - \delta W = \bar{c}(EALc - FL) = 0 \quad \Rightarrow \quad u(x) = cx = \frac{Fx}{EA}
\]
Principle of Virtual Work

• Instead of solving the strong form directly, we want to solve the equation with relaxed requirement (weak form)

• Virtual work - Work resulting from real forces acting through a virtual displacement

• Principle of virtual work - when a system is in equilibrium, the forces applied to the system will not produce any virtual work for arbitrary virtual displacements
  - Balance of linear momentum is force equilibrium \( \nabla \cdot \sigma + \rho \mathbf{f}_b = 0 \)
  - Thus, the virtual work can be obtained by multiplying the force equilibrium equation with a virtual displacement

\[
\overline{W} = \iint_{\Omega} (\nabla \cdot \sigma + \mathbf{f}_b) \cdot \mathbf{u} \, d\Omega
\]

  - If the above virtual work becomes zero for arbitrary \( \mathbf{u} \), then it satisfies the original equilibrium equation in a weak sense

Principle of Virtual Work cont

• PVW \[
\iint_{\Omega} (\sigma_{ij,j} + f_{j}^{b}) \overline{u}_j \, d\Omega = 0 \quad \forall \mathbf{u} \in \mathbb{Z}
\]
  \[
  = \iint_{\Omega} \sigma_{ij,j} \overline{u}_j \, d\Omega + \iint_{\Omega} f_{j}^{b} \overline{u}_j \, d\Omega
  \]
  - Integration-by-parts
  \[
  = \iint_{\Omega} \left[ (\sigma_{ij,j})_i - \sigma_{ij,j} \right] \, d\Omega + \iint_{\Omega} f_{j}^{b} \overline{u}_j \, d\Omega
  \]
  - Divergence Thm
  \[
  = \oint_{\Gamma} n_i \sigma_{ij,j} \overline{u}_j \, d\Gamma - \iint_{\Omega} \sigma_{ij,j} \overline{u}_j \, d\Omega + \iint_{\Omega} f_{j}^{b} \overline{u}_j \, d\Omega
  \]
  - The boundary is decomposed by \( \Gamma = \Gamma^h \cup \Gamma^s \)
    \( \overline{u}_j = 0 \) on \( \Gamma^h \) and \( n_i \sigma_{ij} = t_j \) on \( \Gamma^s \)
  \[
  = \oint_{\Gamma^s} t_j \overline{u}_j \, d\Gamma - \iint_{\Omega} \sigma_{ij,j} \overline{u}_j \, d\Omega + \iint_{\Omega} f_{j}^{b} \overline{u}_j \, d\Omega
  \]
Principle of Virtual Work cont

- Since $\sigma_{ij}$ is symmetric
  \[ \sigma_{ij} \bar{u}_{j,j} = \sigma_{ij} \text{sym}(\bar{u}_{j,j}) = \sigma_{ij} \bar{\varepsilon}_{ij} \]
  \[ \text{sym}(\bar{u}_{j,j}) = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) = \bar{\varepsilon}_{ij} \]

- **Weak Form of BVP**

  \[
  \iiint_{\Omega} \sigma_{ij} \bar{\varepsilon}_{ij} \, d\Omega = \iint_{\Omega} f^b \bar{u}_j \, d\Omega + \int_{\Gamma} t_j \bar{u}_j \, d\Gamma \quad \forall \bar{u} \in \mathbb{Z}
  \]

  Internal virtual work = external virtual work

  Starting point of FEM

- **Symbolic expression**

  \[
  a(u, \bar{u}) = \ell(\bar{u}) \quad \forall \bar{u} \in \mathbb{Z} \quad \longrightarrow \quad [K][d] = [F]
  \]

  - Energy form: \( a(u, \bar{u}) = \iint_{\Omega} \sigma : \varepsilon \, d\Omega \)
  - Load form: \( \ell(\bar{u}) = \iint_{\Omega} \rho \bar{u} \cdot f^b \, d\Omega + \int_{\Gamma} \bar{u} \cdot t \, d\Gamma \)

---

**Example - Heat Transfer Problem**

- **Steady-State Differential Equation**

  \[
  \frac{\partial}{\partial x} \left( k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial T}{\partial y} \right) + Q = 0
  \]

- **Boundary conditions**

  \[
  \begin{cases}
  T = T_0 \text{ on } S_T \\
  q_n = -n_x k_x \frac{dT}{dx} - n_y k_y \frac{dT}{dy} \quad \text{on } S_q
  \end{cases}
  \]

- **Space of kinematically admissible temperature**

  \[
  \mathbb{Z} = \{ \bar{T} \in H^1(\Omega) | \bar{T}(x) = 0, \forall x \in S_T \}
  \]

- **Multiply by virtual temperature, integrate by part, and apply boundary conditions**

  \[
  \iint_{\Omega} \left( k_x \frac{\partial \bar{T}}{\partial x} \frac{\partial T}{\partial x} + k_y \frac{\partial \bar{T}}{\partial y} \frac{\partial T}{\partial y} \right) \, d\Omega = \iint_{\Omega} \bar{T} Q \, d\Omega + \int_{S_q} \bar{T} q_n \, dS_q, \quad \forall \bar{T} \in \mathbb{Z}
  \]
Example – Beam Problem

• Governing DE

\[ EI \frac{d^4 v}{dx^4} = f(x), \quad x \in [0, L] \]

• Boundary conditions for cantilevered beam

\[ v(0) = \frac{dv}{dx}(0) = \frac{d^2 v}{dx^2}(L) = \frac{d^3 v}{dx^3}(L) = 0 \]

• Space of kinematically admissible displacement

\[ \mathcal{Z} = \left\{ \bar{v} \in H^2[0, L] | \bar{v}(0) = \frac{d\bar{v}}{dx}(0) = 0 \right\} \]

• Integrate-by-part twice, and apply BCs

\[ \int_0^L EI \frac{d^2\bar{v}}{dx^2} \frac{d^2v}{dx^2} dx = \int_0^L f\bar{v} dx, \quad \forall \bar{v} \in \mathcal{Z} \]

Difference b/w Strong and Weak Solutions

• The solution of the strong form needs to be twice differentiable

\[ \frac{\partial}{\partial x} \left( k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial T}{\partial y} \right) + Q = 0 \]

• The solution of the weak form requires the first-order derivatives are integrable \( \Rightarrow \) bigger solution space than that of the strong form

\[ \int_{\Omega} \left( k_x \frac{\partial T}{\partial x} + k_y \frac{\partial T}{\partial y} \right) d\Omega \]

• If the strong form has a solution, it is the solution of the weak form

• If the strong form does not have a solution, the weak form may have a natural solution
1.5

FINITE ELEMENT METHOD

Finite Element Approximation

- Difficult to solve a variational equation analytically
- Approximate solution
  - Linear combination of trial functions
  \[ u(x) = \sum_{i=1}^{n} c_i \phi_i(x) \]
  - Smoothness & accuracy depend on the choice of trial functions
  - If the approximate solution is expressed in the entire domain, it is difficult to satisfy kinematically admissible conditions
- Finite element approximation
  - Approximate solution in simple sub-domains (elements)
  - Simple trial functions (low-order polynomials) within an element
  - Kinematically admissible conditions only for elements on the boundary
 Finite Elements

• Types of finite elements

1D 2D 3D

10.1 10.2 0.9

dx dx dx dx

Linear

Quadratic

• Variational equation is imposed on each element.

\[ \int_{0}^{1} f(x) \, dx = \int_{0}^{0.1} f(x) \, dx + \int_{0.1}^{0.2} f(x) \, dx + \cdots + \int_{0.9}^{1} f(x) \, dx \]

Trial Solution

- Solution within an element is approximated using simple polynomials.

\[ u(x) = a_0 + a_1 x, \quad x_i \leq x \leq x_{i+1} \]

- The unknown coefficients, \( a_0 \) and \( a_1 \), will be expressed in terms of nodal solutions \( u(x_i) \) and \( u(x_{i+1}) \).
- Substitute two nodal values
\[
\begin{align*}
  u(x_i) &= u_i = a_0 + a_1 x_i \\
  u(x_{i+1}) &= u_{i+1} = a_0 + a_1 x_{i+1}
\end{align*}
\]
- Express $a_0$ and $a_1$ in terms of $u_i$ and $u_{i+1}$. Then, the solution is approximated by
\[
u(x) = \frac{x_{i+1} - x}{L(e)} u_i + \frac{x - x_i}{L(e)} u_{i+1}
\]
- Solution for Element $e$:
\[
u(x) = N_1(x) u_i + N_2(x) u_{i+1}, \quad x_i \leq x \leq x_{i+1}
\]
- $N_1(x)$ and $N_2(x)$: **Shape Function or Interpolation Function**

**Trial Solution cont.**

- Observations
  - Solution $u(x)$ is interpolated using its nodal values $u_i$ and $u_{i+1}$.
  - $N_1(x) = 1$ at node $x_i$, and $=0$ at node $x_{i+1}$.

- The solution is approximated by **piecewise linear polynomial** and its gradient is constant within an element.

- Stress and strain (derivative) are often averaged at the node.
1D Finite Elements

- 1D BVP \( \frac{d^2 u}{dx^2} + p(x) = 0, \ 0 \leq x \leq 1 \)
  \[ u(0) = 0 \]
  \[ \frac{du}{dx} (1) = 0 \]
  \[ \text{Boundary conditions} \]

- Use PVW \( \int_0^1 \left( \frac{d^2 u}{dx^2} + p \right) \bar{u} \, dx = 0 \)
  \[ Z = \left\{ \bar{u} \in H^{(1)}[0,1] | \bar{u}(0) = 0 \right\} \]

- Integration-by-parts
  \[ \left. \frac{du}{dx} \right|_0^1 - \int_0^1 \frac{du}{dx} \frac{d\bar{u}}{dx} \, dx = -\int_0^1 p\bar{u} \, dx \]
  - This variational equation also satisfies at individual element level

\[ \int_{x_i}^{x_j} \frac{du}{dx} \frac{d\bar{u}}{dx} \, dx = \int_{x_i}^{x_j} p\bar{u} \, dx \]
  \[ \forall \bar{u} \in Z \]

1D Interpolation Functions

- Finite element approximation for one element \( (e) \) at a time

\[ u^{(e)}(x) = u_i N_i(x) + u_{i+1} N_{i+1}(x) = N^{(e)} \cdot d^{(e)} \]

\[ d^{(e)} = \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} \]

\[ N^{(e)} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \]

- Satisfies interpolation condition

\[ u^{(e)}(x_i) = u_i \]
\[ u^{(e)}(x_{i+1}) = u_{i+1} \]

- Interpolation of displacement variation (same with \( u \))

\[ \bar{u}^{(e)}(x) = \bar{u} N_i(x) + \bar{u}_{i+1} N_{i+1}(x) = N^{(e)} \cdot \bar{d}^{(e)} \]

- Derivative of \( u(x) \): differentiating interpolation functions

\[ \frac{du^{(e)}}{dx} = \begin{bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \end{bmatrix} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{L^{(e)}} \\ \frac{1}{L^{(e)}} \end{bmatrix} \begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} = B^{(e)} \cdot d^{(e)} \]
Element-Level Variational Equation

- Approximate variational equation (1) for element (e)

\[
\overline{d}^{(e)^T} \left[ \int_{x_i}^{x_j} B^{(e)^T} B^{(e)} \, dx \right] d^{(e)} = \overline{d}^{(e)^T} \int_{x_i}^{x_j} N^{(e)^T} p(x) \, dx + \overline{d}^{(e)^T} \left\{ \begin{array}{c} -\frac{du}{dx}(x_i) \\ +\frac{du}{dx}(x_{i+1}) \end{array} \right\}
\]

- Must satisfied for all \( \overline{u}^{(e)}(x) \in \mathbb{Z} \)
- If element (e) is not on the boundary, \( \overline{d}^{(e)} \) can be arbitrary

- Element-level variational equation

\[
\begin{bmatrix} \int_{x_i}^{x_j} B^{(e)^T} B^{(e)} \, dx \end{bmatrix} d^{(e)} = \int_{x_i}^{x_j} N^{(e)^T} p(x) \, dx + \left\{ \begin{array}{c} -\frac{du}{dx}(x_i) \\ +\frac{du}{dx}(x_{i+1}) \end{array} \right\}
\]

Assembly

- Need to derive the element-level equation for all elements
- Consider Elements 1 and 2 (connected at Node 2)

\[
\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^{(1)} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{bmatrix}^{(1)} + \left\{ \begin{array}{c} -\frac{du}{dx}(x_1) \\ +\frac{du}{dx}(x_2) \end{array} \right\}
\]

\[
\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}^{(2)} = \begin{bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{bmatrix}^{(2)} + \left\{ \begin{array}{c} -\frac{du}{dx}(x_2) \\ +\frac{du}{dx}(x_3) \end{array} \right\}
\]

- Assembly

\[
\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{21}^{(2)} & k_{22}^{(2)} \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} \end{bmatrix} + \left\{ \begin{array}{c} -\frac{du}{dx}(x_1) \\ 0 \\ +\frac{du}{dx}(x_3) \end{array} \right\}
\]

Vanished unknown term
Assembly cont.

- Assembly of $N_E$ elements ($N_D = N_E + 1$)

$$
\begin{bmatrix}
    k_{11}^{(1)} & k_{12}^{(1)} & 0 & \cdots & 0 \\
    k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & \cdots & 0 \\
    0 & k_{22}^{(2)} & k_{22}^{(2)} + k_{11}^{(2)} & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & k_{21}^{(N_E)} & k_{22}^{(N_E)} \\
\end{bmatrix}_{(N_b \times N_b)}
\begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    \vdots \\
    u_N \\
\end{bmatrix}_{(N_b \times 1)}
\begin{bmatrix}
    f_1^{(1)} \\
    f_2^{(1)} + f_2^{(2)} \\
    f_3^{(2)} + f_3^{(3)} \\
    \vdots \\
    f_N^{(N_E)} \\
\end{bmatrix}_{(N_b \times 1)}
\begin{bmatrix}
    -\frac{du}{dx}(x_1) \\
    0 \\
    0 \\
    \vdots \\
    + \frac{du}{dx}(x_N) \\
\end{bmatrix}_{(N_b \times 1)}
$$

$[K][d] = \{F\}$

- Coefficient matrix $[K]$ is singular; it will become non-singular after applying boundary conditions

Example

- Use three equal-length elements
  $$\frac{d^2 u}{dx^2} + x = 0, \ 0 \leq x \leq 1 \quad u(0) = 0, \ u(1) = 0$$

- All elements have the same coefficient matrix
  $$\left[k^{(e)}\right]_{2 \times 2} = \frac{1}{L^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}, \quad (e = 1, 2, 3)$$

- RHS ($p(x) = x$)

  $\{f^{(e)}\} = \int_{x_i}^{x_{i+1}} p(x) \begin{bmatrix} N_1(x) \\ N_2(x) \end{bmatrix} dx = \frac{1}{L^{(e)}} \int_{x_i}^{x_{i+1}} \left\{ x(x_{i+1} - x) \right\} dx$

  $$= L^{(e)} \begin{bmatrix} \frac{x_i}{3} + \frac{x_{i+1}}{6} \\ \frac{x_i}{6} + \frac{x_{i+1}}{3} \end{bmatrix}, \quad (e = 1, 2, 3)$$
**Example cont.**

- **RHS cont.**
  
  \[
  \begin{aligned}
  f_{1}^{(1)} &= \frac{1}{54} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, & f_{2}^{(1)} &= \frac{1}{54} \begin{bmatrix} 4 \\ 5 \end{bmatrix}, & f_{3}^{(1)} &= \frac{1}{54} \begin{bmatrix} 7 \\ 8 \end{bmatrix}
  \end{aligned}
  \]

- **Assembly**

  \[
  \begin{bmatrix}
  3 & -3 & 0 & 0 \\
  -3 & 3 & 3 & -3 \\
  0 & -3 & 3 & 3 \\
  0 & 0 & -3 & 3
  \end{bmatrix}
  \begin{bmatrix}
  u_{1} \\
  u_{2} \\
  u_{3} \\
  u_{4}
  \end{bmatrix}
  =
  \begin{bmatrix}
  1 \\
  2 \\
  7 \\
  8
  \end{bmatrix}
  \frac{du}{dx} (0)
  \]

- **Apply boundary conditions**

  - Deleting 1st and 4th rows and columns

  \[
  \begin{bmatrix}
  6 & -3 \\
  -3 & 6
  \end{bmatrix}
  \begin{bmatrix}
  u_{2} \\
  u_{3}
  \end{bmatrix}
  =
  \begin{bmatrix}
  1 \\
  9
  \end{bmatrix}
  \begin{bmatrix}
  1 \\
  2
  \end{bmatrix}
  \rightarrow
  u_{2} = \frac{4}{81},
  u_{3} = \frac{5}{81}
  \]

- **EXAMPLE cont.**

- **Approximate solution**

  \[
  u(x) = \begin{cases}
  \frac{4}{27} x, & 0 \leq x \leq \frac{1}{3} \\
  \frac{4}{81} + \frac{1}{27} \left( x - \frac{1}{3} \right), & \frac{1}{3} \leq x \leq \frac{2}{3} \\
  \frac{5}{81} - \frac{5}{27} \left( x - \frac{2}{3} \right), & \frac{2}{3} \leq x \leq 1
  \end{cases}
  \]

- **Exact solution**

  \[
  u(x) = \frac{1}{6} x \left( 1 - x^2 \right)
  \]

  - Three element solutions are poor
  - Need more elements
3D Solid Element

- **Isoparametric mapping**
  - Build interpolation functions on the reference element
  - Jacobian: mapping relation between physical and reference elem.

- **Interpolation and mapping**

\[
\begin{align*}
\mathbf{u}(\xi) &= \sum_{I=1}^{8} N_I(\xi) \mathbf{u}_I \\
\mathbf{x}(\xi) &= \sum_{I=1}^{8} N_I(\xi) \mathbf{x}_I \\
N_I(\xi) &= \frac{1}{8} (1 + \xi I)(1 + \eta I)(1 + \zeta I)
\end{align*}
\]

Same for mapping and interpolation

- **Jacobian matrix**

\[
\mathbf{J} = \frac{\partial \mathbf{x}}{\partial \xi} = \sum_{I=1}^{8} \mathbf{x}_I \frac{\partial N_I(\xi)}{\partial \xi}
\]

- **Derivatives of shape functions**

\[
\begin{bmatrix}
\frac{\partial N_I}{\partial \xi} & \frac{\partial N_I}{\partial \eta} & \frac{\partial N_I}{\partial \zeta}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial N_I}{\partial x_1} & \frac{\partial N_I}{\partial x_2} & \frac{\partial N_I}{\partial x_3}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{\partial x_1}{\partial \xi} & \frac{\partial x_2}{\partial \eta} & \frac{\partial x_3}{\partial \zeta} \\
\frac{\partial x_2}{\partial \xi} & \frac{\partial x_1}{\partial \eta} & \frac{\partial x_3}{\partial \zeta} \\
\frac{\partial x_3}{\partial \xi} & \frac{\partial x_2}{\partial \eta} & \frac{\partial x_1}{\partial \zeta}
\end{bmatrix}
\]

- Jacobian should not be zero anywhere in the element
- Zero or negative Jacobian: mapping is invalid (bad element shape)
3D Solid Element cont.

• Displacement-strain relation

\[ \varepsilon(u) = \sum_{I=1}^{8} B_I u_I \]

\[ \bar{\varepsilon} = \varepsilon(\bar{u}) = \sum_{I=1}^{8} B_I \bar{u}_I \]

\[
\begin{bmatrix}
N_{I,1} & 0 & 0 \\
0 & N_{I,2} & 0 \\
0 & 0 & N_{I,3}
\end{bmatrix}
\]

\[ B_I = \frac{\partial N_I}{\partial x_i} \]

3D Solid Element cont.

• Transformation of integration domain

\[ \int \int \int_{\Omega} d\Omega = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} |J| d\xi d\eta d\zeta \]

• Energy form

\[ a(u, \bar{u}) = \sum_{I=1}^{8} \sum_{J=1}^{8} \bar{u}_I^T \left[ \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} B_I^T D B_J |J| d\xi d\eta d\zeta \right] u_J = \{ \bar{d} \}^T [k] \{ d \} \]

• Load form

\[ l(\bar{u}) = \sum_{I=1}^{8} \bar{u}_I^T \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} N_I(\xi) f^b |J| d\xi d\eta d\zeta = \{ \bar{d} \}^T \{ f \} \]

• Discrete variational equation

\[ \{ \bar{d} \}^T [k] \{ d \} = \{ \bar{d} \}^T \{ f \}, \quad \forall \{ \bar{d} \} \in \mathbb{Z}_h \]
Numerical Integration

• For bar and beam, analytical integration is possible
• For plate and solid, analytical integration is difficult, if not impossible
• Gauss quadrature is most popular in FEM due to simplicity and accuracy
• 1D Gauss quadrature

\[ \int_{-1}^{1} f(\xi) d\xi \approx \sum_{i=1}^{\text{NG}} \omega_i f(\xi_i) \]

- \( \text{NG} \): No. of integ. points; \( \xi_i \): integ. point; \( \omega_i \): integ. weight
- \( \xi_i \) and \( \omega_i \) are chosen so that the integration is exact for \((2*\text{NG} - 1)\)-order polynomial
- Works well for smooth function
- Integration domain is \([-1, 1]\)

Numerical Integration cont.

• Multi-dimensions

\[ \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) d\xi d\eta = \sum_{i=1}^{\text{NG}} \sum_{j=1}^{\text{NG}} \omega_i \omega_j f(\xi_i, \eta_j) \]

\[ \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta, \zeta) d\xi d\eta d\zeta = \sum_{i=1}^{\text{NG}} \sum_{j=1}^{\text{NG}} \sum_{k=1}^{\text{NG}} \omega_i \omega_j \omega_k f(\xi_i, \eta_j, \zeta_k) \]

<table>
<thead>
<tr>
<th>( \text{NG} )</th>
<th>Integration Points ((\xi_i))</th>
<th>Weights ((\omega_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>2.0</td>
</tr>
<tr>
<td>2</td>
<td>\pm 0.5773502692</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>\pm 0.7745966692</td>
<td>\pm 0.5555555556</td>
</tr>
<tr>
<td>4</td>
<td>0.0</td>
<td>\pm 0.8888888889</td>
</tr>
<tr>
<td>5</td>
<td>\pm 0.8611363116</td>
<td>\pm 0.3478546451</td>
</tr>
<tr>
<td>6</td>
<td>\pm 0.3399810436</td>
<td>\pm 0.6521451549</td>
</tr>
<tr>
<td>7</td>
<td>\pm 0.9061798459</td>
<td>\pm 0.2369268851</td>
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<tr>
<td>8</td>
<td>\pm 0.5384693101</td>
<td>\pm 0.4786286705</td>
</tr>
<tr>
<td>9</td>
<td>0.0</td>
<td>\pm 0.5688888889</td>
</tr>
</tbody>
</table>
ELAST3D.m

- A module to solve linear elastic problem using NLFEA.m

- Input variables for ELAST3D.m

<table>
<thead>
<tr>
<th>Variable</th>
<th>Array size</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>ETAN</td>
<td>(6,6)</td>
<td>Elastic stiffness matrix Eq. (1.81)</td>
</tr>
<tr>
<td>UPDATE</td>
<td>Logical variable</td>
<td>If true, save stress values</td>
</tr>
<tr>
<td>LTAN</td>
<td>Logical variable</td>
<td>If true, calculate the global stiffness matrix</td>
</tr>
<tr>
<td>NE</td>
<td>Integer</td>
<td>Total number of elements</td>
</tr>
<tr>
<td>NDOF</td>
<td>Integer</td>
<td>Dimension of problem (3)</td>
</tr>
<tr>
<td>XYZ</td>
<td>(3,NNODE)</td>
<td>Coordinates of all nodes</td>
</tr>
<tr>
<td>LE</td>
<td>(8,NE)</td>
<td>Element connectivity</td>
</tr>
</tbody>
</table>

How to Solve Linear Problem in Nonlinear Code

- Linear matrix solver

\[ [K][d] = \{F\} \iff \{f^{int}\} = \{f^{ext}\} \iff \{f\} = \{f^{ext}\} - \{f^{int}\} = \{0\} \]

  - Construct stiffness matrix and force vector
  - Use LU decomposition to solve for unknown displacement \{d\}

- Nonlinear solver (iterative solver)

  - Assume the solution at iteration \(n\) is known, and \(n+1\) is unknown

\[ \{d^{n+1}\} = \{d^n\} + \{\Delta d\} \quad \text{For linear problem, } \{d^n\} = \{0\} \]

\[ \{f^{n+1}\} = \{f^n\} + \left[ \frac{\partial f}{\partial d} \right] \{\Delta d\} \approx \{0\} \]

\[ \iff \{F\} - [K]\{d^n\} - [K]\{\Delta d\} = 0 \]

\[ \iff [K]\{\Delta d\} = \{F\} \quad \text{Only one iteration!!} \]
function ELAST3D(ETAN, UPDATE, LTAN, NE, NDOF, XYZ, LE)
% MAIN PROGRAM COMPUTING GLOBAL STIFFNESS MATRIX AND RESIDUAL FORCE FOR
% ELASTIC MATERIAL MODELS
%
%global DISPTD FORCE GKF SIGMA
%
% Integration points and weights (2-point integration)
XG=[-0.57735026918963D0, 0.57735026918963D0];
WGT=[1.00000000000000D0, 1.00000000000000D0];
%
% Index for history variables (each integration pt)
INTN=0;
%
% LOOP OVER ELEMENTS, THIS IS MAIN LOOP TO COMPUTE K AND F
for IE=1:NE
% Nodal coordinates and incremental displacements
ELXY=XYZ(LE(IE,:),:);
% Local to global mapping
IDOF=zeros(1,24);
for I=1:8
II=(I-1)*NDOF+1;
IDOF(II:II+2)=(LE(IE,I)-1)*NDOF+1:(LE(IE,I)-1)*NDOF+3;
end
DSP=DISPTD(IDOF);
DSP=reshape(DSP,NDOF,8);
%
% LOOP OVER INTEGRATION POINTS
for LX=1:2, for LY=1:2, for LZ=1:2
E1=XG(LX); E2=XG(LY); E3=XG(LZ);
INTN = INTN + 1;
%
% Determinant and shape function derivatives
 [~, SHPD, DET] = SHAPEL([E1 E2 E3], ELXY);
FAC=WGT(LX)*WGT(LY)*WGT(LZ)*DET;
%
% Strain
DEPS=DSP*SHPD';
DDEPS=[DEPS(1,1) DEPS(2,2) DEPS(3,3) ...
DEPS(1,2)+DEPS(2,1) DEPS(2,3)+DEPS(3,2) DEPS(1,3)+DEPS(3,1)]';
%
% Stress
STRESS = ETAN*DDEPS;
%
% Update stress
if UPDATE
SIGMA(:,INTN)=STRESS;
continue;
end
%
% Add residual force and stiffness matrix
BM=zeros(6,24);
for I=1:8
COL=(I-1)*3+1:(I-1)*3+3;
BM(:,COL)=[SHPD(1,I) 0 0;
0 SHPD(2,I) 0;
0 0 SHPD(3,I);
SHPD(2,I) SHPD(1,I) 0;
0 SHPD(3,I) SHPD(2,I);
SHPD(3,I) 0 SHPD(1,I)];
end
%
% Residual forces
FORCE(IDOF) = FORCE(IDOF) - FAC*BM'*STRESS;
%
% Tangent stiffness
if LTAN
EKF = BM'*ETAN*BM;
GKF(IDOF,IDOF)=GKF(IDOF,IDOF)+FAC*EKF;
end
end, end, end, end
end
function [SF, GDSF, DET] = SHAPEL(XI, ELXY)
%***************************************************************************
% Compute shape function, derivatives, and determinant of hexahedral element
%***************************************************************************
%%
%% XNODE=[-1 1 1 -1 1 -1;
%-1 -1 -1 -1 -1 -1;
%-1 -1 -1 -1 1 1];
QUAR = 0.125;
SF=zeros(8,1);
DSF=zeros(3,8);
for I=1:8
XP = XNODE(1,I);
YP = XNODE(2,I);
ZP = XNODE(3,I);
% XI0 = [1+XI(1)*XP 1+XI(2)*YP 1+XI(3)*ZP];
% SF(I) = QUAR*XI0(1)*XI0(2)*XI0(3);
DSF(1,I) = QUAR*XP*XI0(2)*XI0(3);
DSF(2,I) = QUAR*YP*XI0(1)*XI0(3);
DSF(3,I) = QUAR*ZP*XI0(1)*XI0(2);
end
GJ = DSF*ELXY;
DET = det(GJ);
GJINV=inv(GJ);
GDSF=GJINV*DSF;
end

One Element Tension Example

% One element example
% Nodal coordinates
XYZ=[0 0 0;1 0 0;1 1 0;0 1 0;0 0 1;1 0 1;1 1 1;0 1 1];
% Element connectivity
LE=[1 2 3 4 5 6 7 8];
% External forces [Node, DOF, Value]
EXTFORCE=[5 3 10.0E3; 6 3 10.0E3; 7 3 10.0E3; 8 3 10.0E3];
% Prescribed displacements [Node, DOF, Value]
SDISPT=[1 1 0;1 2 0;1 3 0;2 2 0;2 3 0;3 3 0;4 1 0;4 3 0];
% Material properties
% MID:0(Linear elastic) PROP=[LAMBDA NU]
MID=0;
PROP=[110.747E3 80.1938E3];
% Load increments [Start End Increment InitialFactor FinalFactor]
TIMS=[0.0 1.0 1.0 0.0 1.0];
% Set program parameters
ITRA=30; ATOL=1.0E5; NTOL=6; TOL=1E-6;
% Calling main function
NOUT = fopen('output.txt','w');
NLFEA(ITRA, TOL, ATOL, NTOL, TIMS, NOUT, MID, PROP, EXTFORCE, SDISPT, XYZ, LE);
fclose(NOUT);
One Element Output

**Command line output**

<table>
<thead>
<tr>
<th>Time</th>
<th>Time step</th>
<th>Iter</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000</td>
<td>1.000e+00</td>
<td>2</td>
<td>5.45697e-12</td>
</tr>
</tbody>
</table>

**Contents in output.txt**

TIME = 1.000e+00

**Nodal Displacements**

<table>
<thead>
<tr>
<th>Node</th>
<th>U1</th>
<th>U2</th>
<th>U3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000e+00</td>
<td>0.000e+00</td>
<td>0.000e+00</td>
</tr>
<tr>
<td>2</td>
<td>-5.607e-08</td>
<td>0.000e+00</td>
<td>0.000e+00</td>
</tr>
<tr>
<td>3</td>
<td>-5.607e-08</td>
<td>-5.607e-08</td>
<td>0.000e+00</td>
</tr>
<tr>
<td>4</td>
<td>0.000e+00</td>
<td>-5.607e-08</td>
<td>0.000e+00</td>
</tr>
<tr>
<td>5</td>
<td>-5.494e-23</td>
<td>1.830e-23</td>
<td>1.933e-07</td>
</tr>
<tr>
<td>6</td>
<td>-5.607e-08</td>
<td>4.061e-23</td>
<td>1.933e-07</td>
</tr>
<tr>
<td>7</td>
<td>-5.607e-08</td>
<td>-5.607e-08</td>
<td>1.933e-07</td>
</tr>
<tr>
<td>8</td>
<td>-8.032e-23</td>
<td>-5.607e-08</td>
<td>1.933e-07</td>
</tr>
</tbody>
</table>

**Element Stress**

<table>
<thead>
<tr>
<th>Element</th>
<th>S11</th>
<th>S22</th>
<th>S33</th>
<th>S12</th>
<th>S23</th>
<th>S13</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000e+00</td>
<td>1.091e-11</td>
<td>4.000e+04</td>
<td>-2.322e-13</td>
<td>6.633e-13</td>
<td>-3.317e-12</td>
</tr>
<tr>
<td></td>
<td>0.000e+00</td>
<td>0.000e+00</td>
<td>4.000e+04</td>
<td>-3.980e-13</td>
<td>1.327e-13</td>
<td>-9.287e-13</td>
</tr>
<tr>
<td></td>
<td>-3.638e-12</td>
<td>7.276e-12</td>
<td>4.000e+04</td>
<td>-1.592e-12</td>
<td>-2.123e-12</td>
<td>-3.317e-12</td>
</tr>
<tr>
<td></td>
<td>0.000e+00</td>
<td>0.000e+00</td>
<td>4.000e+04</td>
<td>2.653e-13</td>
<td>-2.123e-12</td>
<td>5.307e-13</td>
</tr>
<tr>
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<td>0.000e+00</td>
<td>0.000e+00</td>
<td>4.000e+04</td>
<td>5.638e-13</td>
<td>3.449e-12</td>
<td>-1.327e-12</td>
</tr>
<tr>
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<td>0.000e+00</td>
<td>0.000e+00</td>
<td>4.000e+04</td>
<td>-1.194e-12</td>
<td>4.776e-12</td>
<td>1.061e-12</td>
</tr>
<tr>
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<td>0.000e+00</td>
<td>0.000e+00</td>
<td>4.000e+04</td>
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<td>3.638e-12</td>
<td>4.000e+04</td>
<td>-5.307e-13</td>
<td>3.715e-12</td>
<td>1.061e-12</td>
</tr>
</tbody>
</table>

*** Successful end of program ***