CHAP 4.4

Hyperelastic Material

EGM 6352 (Fall 2008)
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Goals

- Understand the definition of hyperelastic material
- Understand strain energy density function and how to use it to obtain stress
- Understand the role of invariants in hyperelasticity
- Understand how to impose incompressibility
- Understand mixed formulation and perturbed Lagrangian formulation
- Understand linearization process when strain energy density is written in terms of invariants

What Is Hyperelasticity?

- Hyperelastic material - stress-strain relationship derives from a strain energy density function
  - Stress is a function of total strain (independent of history)
  - Depending on strain energy density, different names are used, such as Mooney-Rivlin, Ogden, Yeoh, or polynomial model
- Generally comes with incompressibility
  - The volume preserves during large deformation
  - Mixed formulation – completely incompressible hyperelasticity
  - Penalty formulation - nearly incompressible hyperelasticity
- Example: rubber, biological tissues
  - non-linearly elastic, isotropic, incompressible and generally independent of strain rate
- Hypoelastic material: relation is given in terms of stress and strain rates

Strain Energy Density

- We are interested in isotropic material
  - Material frame indifference: no matter what coordinate system is chosen, the response of the material is identical
  - The components of a deformation tensor depends on coord. system
  - Three invariants of **C** is independent of coord. system
- Invariants of *C*

$$I_1 =$$

$$I_2 =$$

$$I_3 =$$

  - In order to be material frame indifferent, material properties must be expressed using invariants
  - For incompressibility, $I_3 = 1$

Strain Energy Density cont.

- Strain Energy Density Function
  - Must be zero when **C** = **1**, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$$W(I_1, I_2, I_3) = \sum_{m+n+k=1}^{\infty} A_{mnk}(I_1 - 3)^m (I_2 - 3)^n (I_3 - 1)^k$$

  - For incompressible material

$$W(I_1, I_2) = \sum_{m+n=1}^{\infty} A_{mn}(I_1 - 3)^m (I_2 - 3)^n$$

  - Ex: Neo-Hookean model
  - Mooney-Rivlin model

Example – St. Vernant Kirchhoff Material

- Show that St. Vernant-Kirchhoff material has the following strain energy density

$$W(\mathbf{E}) = \frac{\lambda}{2}\left[\operatorname{tr}(\mathbf{E})\right]^2 + \mu \operatorname{tr}(\mathbf{E}^2)$$

$$\mathbf{S} = \frac{\partial W(\mathbf{E})}{\partial \mathbf{E}} =$$

- First term

$$\operatorname{tr}(\mathbf{E}) = \mathbf{1} : \mathbf{E} \qquad \frac{\partial \operatorname{tr}(\mathbf{E})}{\partial \mathbf{E}} = \mathbf{1}$$

$$\lambda \operatorname{tr}(\mathbf{E}) \frac{\partial \operatorname{tr}(\mathbf{E})}{\partial \mathbf{E}} =$$

- Second term

$$\frac{\partial E_{ij} E_{ji}}{\partial E_{kl}} = \delta_{ik}\delta_{jl}E_{ji} + E_{ij}\delta_{jk}\delta_{il} =$$

Example – St. Vernant Kirchhoff Material cont.

- Thus

$$\mathbf{S} = \lambda \mathrm{tr}(\mathbf{E}) \frac{\partial \mathrm{tr}(\mathbf{E})}{\partial \mathbf{E}} + \mu \frac{\partial \mathrm{tr}(\mathbf{E}^2)}{\partial \mathbf{E}}$$
$$= \lambda (\mathbf{1} \otimes \mathbf{1}) : \mathbf{E} + 2\mu \mathbf{E}$$
$$=$$

Nearly Incompressible Hyperelasticity

- Incompressible material
  - Cannot calculate stress from strain. Why?
- Nearly incompressible material
  - Many material show nearly incompressible behavior
  - We can use the bulk modulus to model it
- Using I_1 and I_2 enough for incompressibility?
  - No, I_1 and I_2 actually changes under hydrostatic deformation
  - We will use reduced invariants: J_1, J_2, and J_3
- Will J_1 and J_2 be constant under dilatation?

Example – Hydrostatic Tension

$$\begin{cases} x_1 = \alpha X_1 \\ x_2 = \alpha X_2 \\ x_3 = \alpha X_3 \end{cases} \qquad \mathbf{F} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{bmatrix}$$

- Invariants

I_1 and I_2 are not constant

- Reduced invariants

$$J_1 = I_1 I_3^{-1/3} = 3$$
$$J_2 = I_2 I_3^{-2/3} = 3$$
$$J_3 = I_3^{1/2} = \alpha^3$$

I_1 and I_2 are constant

Strain Energy Density

- Using reduced invariants
  - $W_1(J_1, J_2)$: Distortional strain energy density
  - $W_2(J_3)$: Dilatational strain energy density
- The second terms is related to nearly incompressible behavior
  - K: bulk modulus $= \lambda + \frac{2}{3}\mu$ for linear elastic material

Mooney-Rivlin Material

- Most popular model

$$W(J_1, J_2, J_3) = W_1(J_1, J_2) + W_2(J_3)$$

$$=$$

  - Shear modulus ~ $2(A_{10} + A_{01})$
  - Young's modulus ~ $6(A_{10} + A_{01})$ (3D) or $8(A_{10} + A_{01})$ (2D)
  - Bulk modulus = K
- Hydrostatic pressure
  - Numerical instability for large K (volumetric locking)

Mooney-Rivlin Material cont.

- Second P-K stress

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = \frac{\partial W}{\partial J_1}\frac{\partial J_1}{\partial \mathbf{E}} + \frac{\partial W}{\partial J_2}\frac{\partial J_2}{\partial \mathbf{E}} + \frac{\partial W}{\partial J_3}\frac{\partial J_3}{\partial \mathbf{E}}$$

$$= A_{10}J_{1,\mathbf{E}} + A_{01}J_{2,\mathbf{E}} + K(J_3 - 1)J_{3,\mathbf{E}}$$

$$a_{,\mathbf{E}} = \frac{\partial a}{\partial \mathbf{E}}$$

  - Use chain rule of differentiation

$$J_{1,\mathbf{E}} = (I_3^{-1/3})I_{1,\mathbf{E}} - \tfrac{1}{3}I_1(I_3^{-4/3})I_{3,\mathbf{E}}$$

$$J_{2,\mathbf{E}} = (I_3^{-2/3})I_{2,\mathbf{E}} - \tfrac{2}{3}I_2(I_3^{-5/3})I_{3,\mathbf{E}}$$

$$J_{3,\mathbf{E}} = \tfrac{1}{2}(I_3^{-1/2})I_{3,\mathbf{E}}$$

$$I_{1,\mathbf{E}} = 2\mathbf{1}$$

$$I_{2,\mathbf{E}} = 4(1 + \mathrm{tr}\mathbf{E})\mathbf{1} - 4\mathbf{E}$$

$$I_{3,\mathbf{E}} = (2 + 4\mathrm{tr}\mathbf{E})\mathbf{1} - 4\mathbf{E} + [\tfrac{9}{4}e_{imn}e_{jrs}E_{mr}E_{ns}]$$

Example

- Show $I_{1,\mathbf{E}} = 2\mathbf{1}, \quad I_{2,\mathbf{E}} = 2(I_1\mathbf{1} - \mathbf{C}), \quad I_{3,\mathbf{E}} = 2I_3\mathbf{C}^{-1}$
- Let $\bar{I}_1 = \text{tr}(\mathbf{C}), \quad \bar{I}_2 = \frac{1}{2}\text{tr}(\mathbf{CC}), \quad \bar{I}_3 = \frac{1}{3}\text{tr}(\mathbf{CCC})$
- Then $I_1 = \bar{I}_1, \quad I_2 = \frac{1}{2}\bar{I}_1^{\,2} - \bar{I}_2, \quad I_3 = \bar{I}_3 + \frac{1}{6}\bar{I}_1^{\,3} - \bar{I}_1\bar{I}_2$
- Derivatives

$$\frac{\partial \bar{I}_1}{\partial C_{ij}} = \delta_{ij}, \quad \frac{\partial \bar{I}_2}{\partial C_{ij}} = C_{ji}, \quad \frac{\partial \bar{I}_3}{\partial C_{ij}} = C_{jk}C_{ki}$$

$$\frac{\partial I_1}{\partial C_{ij}} = \delta_{ij}, \quad \frac{\partial I_2}{\partial C_{ij}} = I_1\delta_{ij} - C_{ji}, \quad \frac{\partial \bar{I}_3}{\partial C_{ij}} = I_3C_{ji}^{-1}$$

and

Mixed Formulation

- Using bulk modulus often causes instability
  - Selectively reduced integration (Full integration for deviatoric part, reduced integration for dilatation part)
- Mixed formulation: Independent treatment of pressure
  - Pressure p is additional unknown (pure incompressible material)
  - Advantage: No numerical instability
  - Disadvantage: system matrix is not positive definite
- Perturbed Lagrangian formulation
  - Second term make the material nearly incompressible and the system matrix positive definite

Variational Equation

- Stress calculation

$$\mathbf{S} = A_{10}J_{1,\mathbf{E}} + A_{01}J_{2,\mathbf{E}} + pJ_{3,\mathbf{E}}$$

- Variation of strain energy density

$$\overline{W} = W_{,\mathbf{E}}\bar{\mathbf{E}} + W_{,p}\bar{p}$$

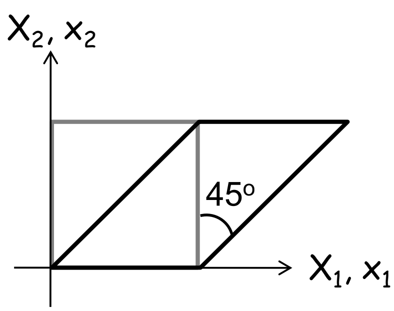
$$=$$

- Introduce a vector of unknowns: $\mathbf{r} = (\mathbf{u}, p)$

$$a_0(\mathbf{r},\bar{\mathbf{r}}) = \iint_{\Omega_0} \left[\mathbf{S} : \bar{\mathbf{E}} + \bar{p}H \right] d\Omega$$

Example – Simple Shear

- Calculate 2nd P-K stress for the simple shear deformation
  - material properties (A_{10}, A_{01}, K)



$$\mathbf{F} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$I_1 = 3, \quad I_2 = 1, \quad I_3 = 1$$

$$I_{1,\mathbf{E}} = 2\mathbf{1}$$

$$I_{2,\mathbf{E}} = 2(I_1\mathbf{1} - \mathbf{C}) = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$$

$$I_{3,\mathbf{E}} = 2I_3\mathbf{C}^{-1} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$$

Example – Simple Shear cont.

$$J_{1,E} = I_{1,E} - I_{3,E} = \begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix}$$

$$J_{2,E} = I_{2,E} - \tfrac{2}{3} I_{3,E} = \frac{1}{3}\begin{bmatrix} 4 & -2 \\ -2 & 0 \end{bmatrix}$$

$$J_{3,E} = \tfrac{1}{2} I_{3,E} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\mathbf{S} = A_{10} J_{1,E} + A_{01} J_{2,E} + K(J_3 - 1) J_{3,E}$$

$$= A_{10}\begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix} + \frac{A_{01}}{3}\begin{bmatrix} 4 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -2A_{10} + \frac{4}{3}A_{01} & 2A_{10} - \frac{2}{3}A_{01} \\ 2A_{10} - \frac{2}{3}A_{01} & 0 \end{bmatrix}$$

Note: S_{11} is not zero

Implementation

- Given: $\{\mathbf{E}\} = \{E_{11}, E_{22}, E_{33}, 2E_{12}\}^T$, $\{p\}$, (A_{10}, A_{01})

$$\{\mathbf{1}\} = \{1 \quad 1 \quad 1 \quad 0\}^T \qquad \{\mathbf{C}\} = 2\{\mathbf{E}\} + \{\mathbf{1}\}$$

$$I_1 = C_1 + C_2 + C_3$$
$$I_2 = C_1C_2 + C_1C_3 + C_2C_3 - C_4C_4$$
$$I_3 = (C_1C_2 - C_4C_4)C_3$$

$$\{I_{1,E}\} = 2\{1 \quad 1 \quad 1 \quad 0\}$$
$$\{I_{2,E}\} = 2\{C_2 + C_3 \quad C_3 + C_1 \quad C_1 + C_2 \quad -C_4\}$$
$$\{I_{3,E}\} = 2\{C_2C_3 \quad C_3C_1 \quad C_1C_2 - C_4C_4 \quad -C_3C_4\}$$
$$\{J_{1,E}\} = I_3^{-1/3}\{I_{1,E}\} - \tfrac{1}{3} I_1 I_3^{-4/3}\{I_{3,E}\}$$
$$\{J_{2,E}\} = I_3^{-2/3}\{I_{2,E}\} - \tfrac{2}{3} I_2 I_3^{-5/3}\{I_{3,E}\}$$
$$\{J_{3,E}\} = \tfrac{1}{2} I_3^{-1/2}\{I_{3,E}\},$$

$$\{\mathbf{S}\} = A_{10}\{J_{1,E}\} + A_{01}\{J_{2,E}\} + p\{J_{3,E}\}$$

For penalty method, use $K(J_3 - 1)$ instead of p

Linearization

- Stress increment

$$\Delta\mathbf{S} = \mathrm{W}_{,\mathbf{E},\mathbf{E}} : \Delta\mathbf{E} + \mathrm{W}_{,\mathbf{E},p}\Delta p$$
$$= \boldsymbol{C} : \Delta\mathbf{E} + J_{3,\mathbf{E}}\Delta p$$

- Linearized energy form

$$a_0^*(\mathbf{r};\Delta\mathbf{r},\bar{\mathbf{r}}) = \iint_{\Omega_0}\left[\bar{\mathbf{E}} : (\boldsymbol{C} : \Delta\mathbf{E} + J_{3,\mathbf{E}}\Delta p) + \mathbf{S} : \Delta\bar{\mathbf{E}}\right]\mathrm{d}\Omega$$
$$+ \iint_{\Omega_0}\bar{p}\left(J_{3,\mathbf{E}} : \Delta\mathbf{E} - \frac{\Delta p}{K}\right)\mathrm{d}\Omega$$

Summary

- Hyperelastic material: strain energy density exists with incompressible constraint
- In order to be material frame indifferent, material properties must be expressed using invariants
- Numerical instability (volumetric locking) can occur when large bulk modulus is used for incompressibility
- Mixed formulation is used for purely incompressibility (additional pressure variable, non-PD tangent stiffness)
- Perturbed Lagrangian formulation for nearly incompressibility (reduced integration for pressure term)