CHAP 5.1

One-Dimensional Elastoplastic Analysis

EGM 6352 (Fall 2008)

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Goals

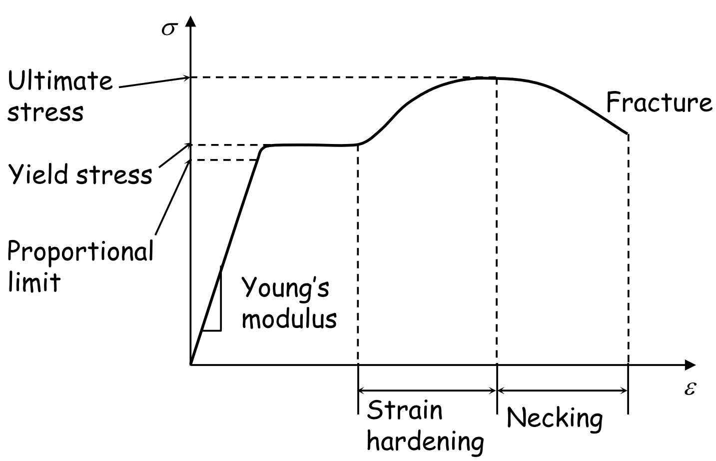
- Understand difference between elasticity and plasticity
- Learn basic elastoplastic model
- Learn different hardening models
- Understand different moduli used in 1D elastoplasticity
- Learn how to calculate plastic strain when total strain increment is given
- Learn state determination for elastoplastic material

Plasticity

- Elasticity – A material deforms under stress, but then returns to its original shape when the stress is removed
- Plasticity - deformation of a material undergoing **non-reversible changes of shape** in response to applied forces
  - Plasticity in metals is usually a consequence of **dislocations**
  - Rough nonlinearity
- Found in most metals, and in general is a good description for a large class of materials
- **Perfect plasticity** – a property of materials to undergo irreversible deformation without any increase in stresses or loads
- **Hardening** - need increasingly higher stresses to result in further plastic deformation

Behavior of a Ductile Material

Terms	Explanation
Proportional limit	The greatest stress for which the stress is still proportional to the strain
Elastic limit	The greatest stress without resulting in any permanent strain on release of stress
Young's Modulus	Slope of the linear portion of the stress-strain curve
Yield stress	The stress required to produce 0.2% plastic strain
Strain hardening	A region where more stress is required to deform the material
Ultimate stress	The maximum stress the material can resist
Necking	Cross section of the specimen reduces during deformation

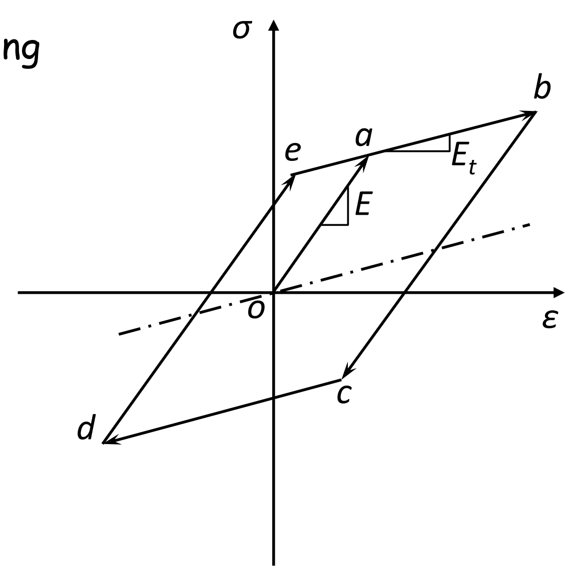


Elastoplasticity

- Most metals have both elastic and plastic properties
  - Initially, the material shows elastic behavior
  - After yielding, the material becomes plastic
  - By removing loading, the material becomes elastic
- We will assume **small (infinitesimal) deformation** case
  - Elastic and plastic strain can be **additively decomposed** by
  - Strain energy density exists in terms of elastic strain
  - Stress is related to the elastic strain, not the plastic strain
- The plastic strain will be considered as an internal variable, which evolves according to plastic deformation

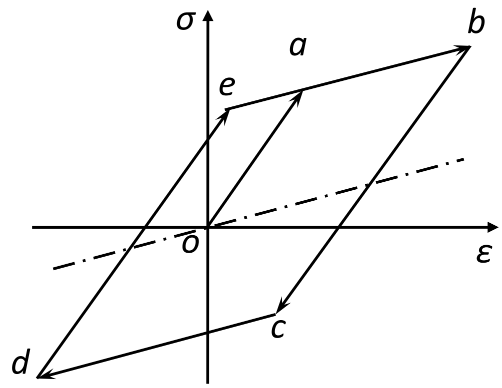
1D Elastoplasticity

- Idealized elastoplastic stress-strain behavior
  - Initial elastic behavior with slope until yield stress σ_Y (line *o–a*)
  - After yielding, the plastic phase with slope (line *a–b*).
  - Upon removing load, elastic unloading with slope E (line b-c)
  - Loading in the opposite direction, the material will eventually yield in that direction (point *c*)
  - – more force is required to continuously deform in the plastic region

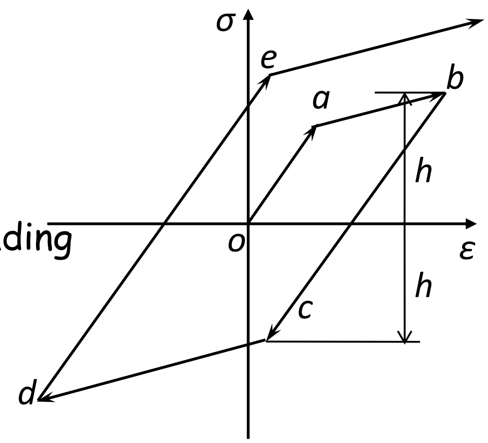


Work Hardening Models

- Kinematic hardening
  -
  - Center of the elastic region moves parallel to the work hardening line
  - bc = de = 2oa
  - Use the center of elastic domain as an evolution variable

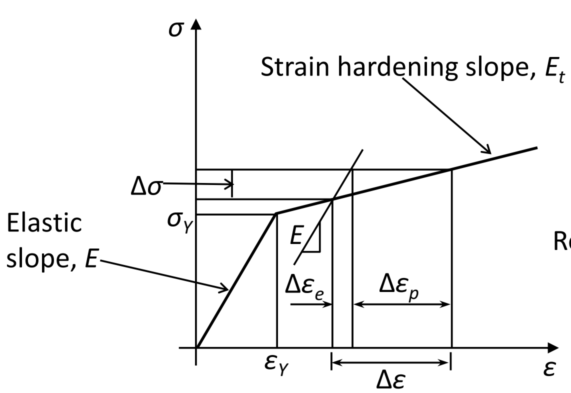


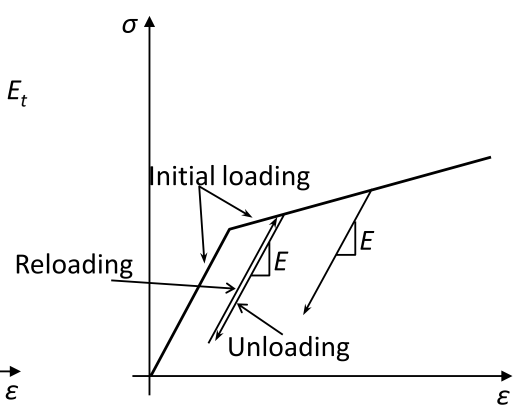
- Isotropic hardening
  -
  - The yield stress for the reversed loading is equal to the previous yield stress
  - Use plastic strain as an evolution variable



Elastoplastic Analysis

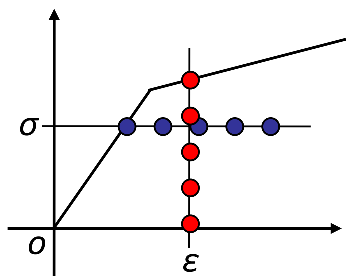
- Additive decomposition
  - Only elastic strain contributes to stress (but we don't know how much of the total strain corresponds to the elastic strain)
  - Let's consider an increment of strain: $\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$
  - Elastic strain increases stress by $\Delta\sigma = E\Delta\varepsilon_e$
  - Elastic strain disappears upon removing loads or changing direction





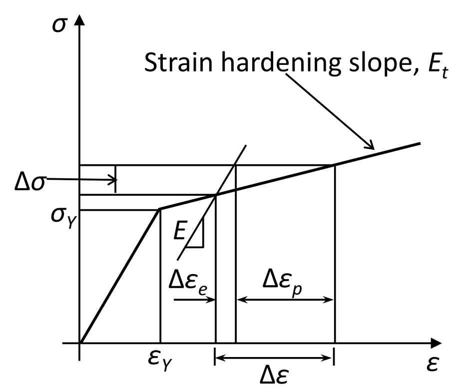
Elastoplastic Analysis cont.

- Additive decomposition (continue)
  - Plastic strain remains constant during unloading
  - The effect of load-history is stored in the plastic strain
  - The yield stress is determined by the magnitude of plastic strain
  - **Decomposing elastic and plastic part of strain is an important part of elastoplastic analysis**
- For given stress σ, strain cannot be determined.
  - Complete history is required (path- or history-dependent)
  - History is stored in evolution variable (plastic strain)



Analysis Procedure

- Strain increment $\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$
- Stress increment $\Delta\sigma = E\Delta\varepsilon_e$
- **Plastic modulus**
- Relation between moduli



$$\frac{\Delta\sigma}{E_t} = \frac{\Delta\sigma}{E} + \frac{\Delta\sigma}{H} \quad \Rightarrow \quad \frac{1}{E_t} = \frac{1}{E} + \frac{1}{H}$$

$$H = \frac{EE_t}{E - E_t} \qquad E_t = \frac{EH}{E + H} = E\left(1 - \frac{E}{E + H}\right)$$

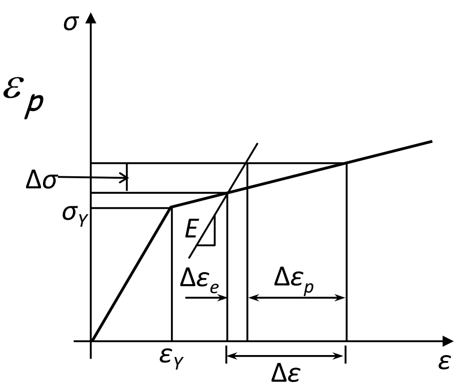
Analysis Procedure cont.

- Analysis is performed with a given incremental strain
  - N-R iteration will provide $\Delta\mathbf{u} \Rightarrow \Delta\varepsilon$
  - But, we don't know $\Delta\varepsilon_e$ or $\Delta\varepsilon_p$
- When the material is in the initial elastic range, regular elastic analysis procedure can be used
- When the material is in the plastic range, we have to **determine incremental plastic strain**

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p = \frac{\Delta\sigma}{E} + \Delta\varepsilon_p = \frac{H\Delta\varepsilon_p}{E} + \Delta\varepsilon_p$$

$$= \Delta\varepsilon_p\left(\frac{H}{E} + 1\right)$$

$$\Rightarrow$$



1D Finite Element Analysis Procedure

- 1D governing equation: $\frac{d}{dx}\left(A\sigma\right) + q = 0$ q = force/unit length, A = cross-section area
  - Stress-strain relationship is incremental (not total; hypoelasticity)
- Discretization in load (time) steps
  - Let the solution is converged up to load step n and n+1 is current
  - Since all states at load step n is known, we need to calculate increments of stress and strain
  - We want to make the DE satisfy at load step n+1

1D Finite Element Analysis Procedure

- Governing equation at load step n+1

$$\frac{d}{dx}(A\sigma^{n+1}) + q^{n+1} = 0$$

$$\Rightarrow \frac{d}{dx}(A\sigma^{n}) + \frac{d}{dx}(A\Delta\sigma) + q^{n} + \Delta q = 0$$

  - Since the equilibrium satisfied at load step n

- Incremental stress-strain relationship is linear

$$C_{ep} = \begin{cases} E & \text{elastic} \\ E_t & \text{plastic} \end{cases}$$

1D Finite Element Analysis Procedure cont.

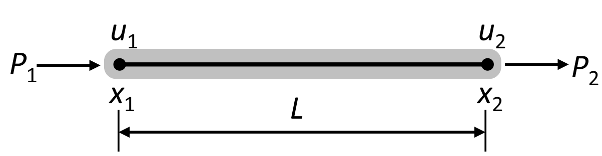
- Incremental DE

Similar to elastic bar equation

  - Apply the principle of virtual work
  - We want to solve it using 1D bar element

- Interpolation $\Delta u(x) = [N_1 \; N_2]\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix} = \mathbf{N} \cdot \Delta\mathbf{d}$

$$\Delta\varepsilon = \frac{d}{dx}\left(\Delta u\right) = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix} = \mathbf{B} \cdot \Delta\mathbf{d}$$



1D Finite Element Analysis Procedure cont.

- Element matrix equation

Uniformly distributed load

$$\frac{AC_{ep}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix} = \frac{\Delta q L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \end{Bmatrix}$$

Nodal forces

- State determination
  - Follow load history (hardening, loading & unloading, yielding, etc)
  - We will discuss it in detail
- Internal forces

$$\mathbf{r}_I = A\int_{x_1}^{x_2} \mathbf{B}^T \sigma \, dx = A\int_{x_1}^{x_2} \frac{1}{L}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \sigma \, dx =$$

State Determination

- How to determine stress
  - Given: strain increment ($\Delta\varepsilon$) and all variables in load step n

1. Computer the current yield stress

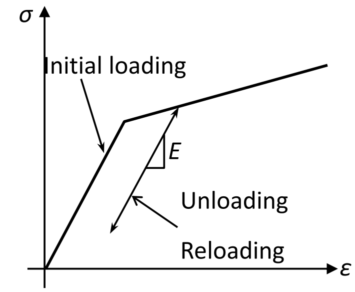
2. Elastic predictor

Assume incremental strain is elastic

3. Check stress status
   1. Previously elastic state
   2. Previously plastic state

State Determination cont.

- Previously elastic state (2 possible cases)



1. If $\sigma^{tr} \leq \sigma_Y^n$, material is still elastic

   Either initial elastic region or unloading

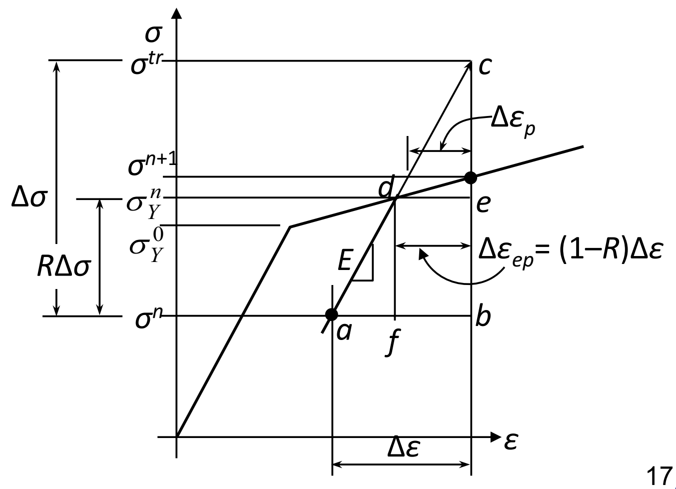
2. If $\sigma^{tr} > \sigma_Y^n$, yielding (transition from elastic to plastic)

   Introduce strain ratio R

   $$\frac{\Delta\varepsilon}{\Delta\sigma} = \frac{(1-R)\Delta\varepsilon}{\sigma^{tr} - \sigma_Y^n}$$

   $\Rightarrow$

   Elastic portion of strain



State Determination cont.

- Previously elastic state (2 possible cases) cont.
  - Elastoplastic strain

    Both elastic and plastic strains due to hardening

  - Plastic strain

$$\Delta\varepsilon_p = \frac{\Delta\varepsilon_{ep}}{1 + H / E}$$

Consider $\Delta\varepsilon_{ep}$ as $\Delta\varepsilon$

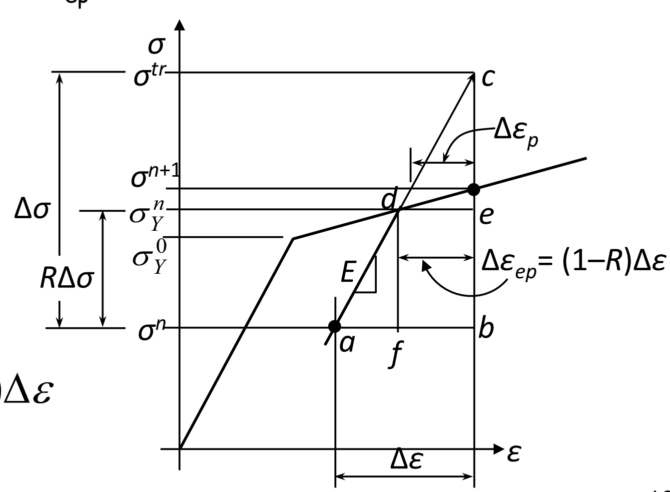
$$\Delta\varepsilon_p = \frac{1-R}{1 + H / E}\Delta\varepsilon$$

  - Stress calculation

$$\sigma^{n+1} = \sigma^n + E\Delta\varepsilon_e + E_t\Delta\varepsilon_{ep}$$

$$= \sigma^n + R\Delta\sigma + E_t(1-R)\Delta\varepsilon$$

$$= \sigma^n +$$

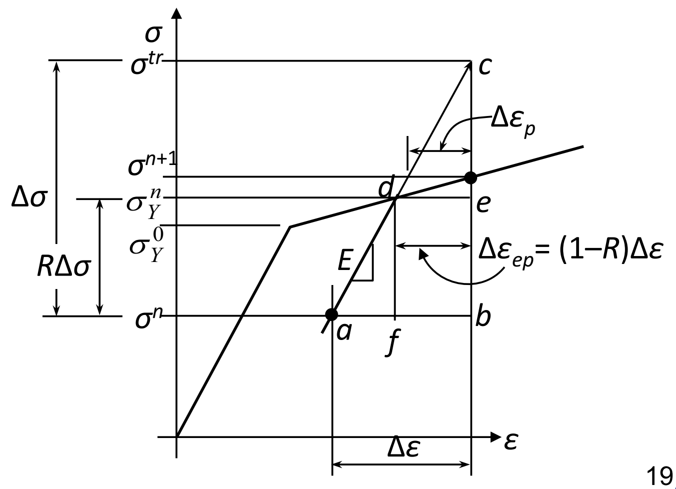


State Determination cont.

- Stress calculation cont.

$$\sigma^{n+1} = \sigma^n + \left[RE + (1-R)E_t \right] \Delta\varepsilon$$
$$= \sigma^n + E\Delta\varepsilon - (1-R)(E - E_t)\Delta\varepsilon$$

- Algorithm
  1) Elastic trial
  2) Plastic return mapping



State Determination cont.

- Stress-strain relation
  - Only incremental relationship is possible (**hypoelasticity**)
  - Initially we started with

(1)
$$C_{ep} = \begin{cases} E & \text{elastic} \\ E_t & \text{plastic} \end{cases}$$

  - But, from state determination algorithm, we have

(2)

  - (1) is called
  - (2) is called
  - (1) and (2) are different because the strain increment has both elastic and elastoplastic parts
  - (2) should be used for N-R iteration to have quadratic convergence

State Determination cont.

- Previously plastic state (2 possible cases)

1. $\sigma^{tr} > \sigma_y^n$: Continuous yielding
   - Use same algorithm with R = 0
2. $\sigma^{tr} \le \sigma_y^n$: Unloading
   - Use trial state as a final stress

- Actual algorithm is more complicated because unloading can yield in the opposite direction and hardening can be isotropic or kinematic

Algorithm for Isotropic Hardening

- Given: $\Delta\varepsilon, E, H, \sigma_y^0, \sigma^n, \varepsilon_p^n$, YSTATE (0 or 1)

1. Trial state $\Delta\sigma = E\Delta\varepsilon, \quad \sigma^{tr} = \sigma^n + \Delta\sigma, \quad \sigma_y^n = \sigma_y^0 + H\varepsilon_p^n$
2. If YSTATE = 0 (elastic)
   a. If $\left|\sigma^{tr}\right| \le \sigma_y^n$ (remain elastic): $\sigma^{n+1} = \sigma^{tr}$; exit
   b. else (become plastic): YSTATE = 1

$$R = 1 - \frac{\left|\sigma^{tr}\right| - \sigma_y^n}{\left|\Delta\sigma\right|}$$

3. If YSTATE = 1 (plastic)
   a. If $\sigma^n\Delta\sigma < 0$ (unloading): YSTATE = 0, $\sigma^{n+1} = \sigma^{tr}$; exit
   b. else (continue to yield): R = 0
4. Update stress and plastic strain

$$\sigma^{n+1} = \sigma^n + R\Delta\sigma + \frac{EH}{E+H}(1-R)\Delta\varepsilon \qquad \varepsilon_p^{n+1} = \varepsilon_p^n + \frac{1-R}{1+H/E}\left|\Delta\varepsilon\right|$$

Algorithm for Kinematic Hardening

- Given: YSTATE, $\Delta\varepsilon, E, H, \sigma_Y^0, \sigma^n, \varepsilon_p^n$

1. Trial state: $\Delta\sigma = E\Delta\varepsilon \qquad \sigma^{tr} = \sigma^n + \Delta\sigma$

   $\sigma_Y^{\max} = \sigma_Y^0 + H\varepsilon_p \qquad \sigma_Y^{\min} = \sigma_Y^{\max} - 2\sigma_Y^0$

2. If YSTATE = 0
   a. If $\sigma_Y^{\min} \le \sigma^{tr} \le \sigma_Y^{\max}$ (elastic): $\sigma^{n+1} = \sigma^{tr}$; exit
   b. else (become plastic): YSTATE = 1;

      If $\sigma^{tr} > \sigma_Y^{\max}$, then $R = 1 - \dfrac{|\sigma^{tr} - \sigma_Y^{\max}|}{|\Delta\sigma|}$; else $R = 1 - \dfrac{|\sigma^{tr} - \sigma_Y^{\min}|}{|\Delta\sigma|}$

3. If YSTATE = 1 (plastic)
   a. If $\sigma^n\Delta\sigma < 0$ (unloading): YSTATE = 0, $\sigma^{n+1} = \sigma^{tr}$; exit
   b. else (continue to yield): R = 0

4. Update stress and plastic strain

$$\sigma^{n+1} = \sigma^n + R\Delta\sigma + \frac{EH}{E+H}(1-R)\Delta\varepsilon \qquad \varepsilon_p^{n+1} = \varepsilon_p^n + \frac{1-R}{1+H/E}|\Delta\varepsilon|$$

Ramberg-Osgood Model

- Ramberg-Osgood plasticity model
  - So far we discussed linear hardening plasticity
  - The plasticity model is complicated due to loading/unloading history even if a simple hardening model is used
  - However, if we know that the problem has only a proportional loading, then the stress-strain relation can be given in **total form**
  - This model should be used with caution because even if the applied load is proportional, stress at a point may not be proportional
- Constitutive relation
  - It is a total form
  - exponential hardening
  - Smooth curve

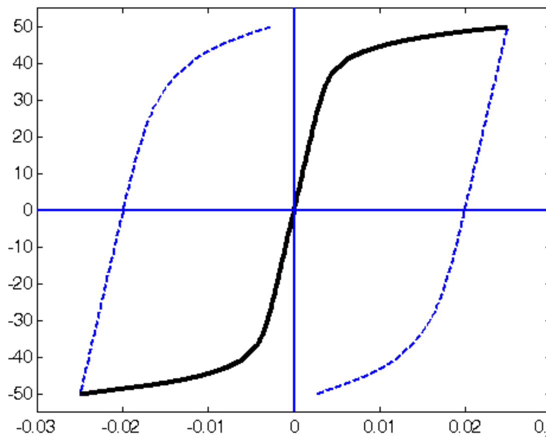
Ramberg-Osgood Model

- Incremental stress-strain relation
  - Differentiate the constitutive relation

$$\Delta\sigma = C_{ep}\Delta\varepsilon$$

- Modification for unloading
  - Unloading occurs at $(\sigma_0, \varepsilon_0)$

$$\varepsilon - \varepsilon_0 = \frac{\sigma - \sigma_0}{E} + \frac{3}{7}\frac{\sigma_y + |\sigma_0|}{E}\left(\frac{\sigma - \sigma_0}{\sigma_y + |\sigma_0|}\right)^n$$



Summary

- Plastic deformation depends on load-history and its information is stored in plastic strain
- Stress only depends on elastic strain
- Isotropic hardening increases the elastic domain, while kinematic hardening maintain the size of elastic domain but moves the center of it
- Major issue in elastoplastic analysis is to decompose the strain into elastic and plastic parts
- Algorithmic tangent stiffness is consistent with the state determination algorithm
- State determination is composed of (a) elastic trial and (b) plastic return mapping