EGM6352 Homework #3 Due: 10/27/08 Name:_______________

1. Consider a deformation of a rectangular bar as shown in the figure. The deformed geometry is given as

$$\begin{cases} x_1 = \alpha X_1 \\ x_2 = \beta X_2 \\ x_3 = \beta X_3 \end{cases}$$

When the material is incompressible, Mooney-Rivlin hyperelastic material with $A_{10} = 80 \text{ MPa}$ and $A_{01} = 20 \text{ MPa}$, write the expression of S_{11} component of the second Piola-Kirchhoff stress as a function of α. In addition, write the expression of σ_{11} of the Cauchy stress as a function of α. Plot S_{11} and σ_{11} in the range of $\alpha = [0.7\ 1.5]$.

Solution: For given deformation, the deformation gradient and Cauchy-Green deformation tensor can be calculated as

$$\mathbf{F} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^2 \end{bmatrix}$$

The relation between α and β can be obtained from incompressibility:

$$\det \mathbf{F} = \alpha\beta^2 = 1 \quad \Rightarrow \quad \beta = \alpha^{-1/2}$$

The three invariants of the deformation tensor can be obtained as

$$\begin{aligned} I_1 &= \alpha^2 - 2\alpha^{-1} \\ I_2 &= 2\alpha + \alpha^{-2} \\ I_3 &= 1 \end{aligned}$$

The reduced invariants become

$$\begin{aligned} J_1 &= I_1 I_3^{-1/3} = \alpha^2 - 2\alpha^{-1} \\ J_2 &= I_2 I_3^{-2/3} = 2\alpha + \alpha^{-2} \\ J_3 &= I_3^{-1/2} = 1 \end{aligned}$$

In order to calculate stress, we need to differentiate the reduced invariants with respect to strain

$$I_{1,\mathbf{E}} = 2\mathbf{1} = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_{2,\mathbf{E}} = 2(I_1\mathbf{1} - \mathbf{C}) = 2\begin{bmatrix} 2\alpha^{-1} & 0 & 0 \\ 0 & \alpha^2 + \alpha^{-1} & 0 \\ 0 & 0 & \alpha^2 + \alpha^{-1} \end{bmatrix}$$

$$I_{3,\mathbf{E}} = 2I_3\mathbf{C}^{-1} = 2\begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

The derivatives of the reduced invariants become

$$J_{1,\mathbf{E}} = I_3^{-1/3}I_{1,\mathbf{E}} - \frac{1}{3}I_1I_3^{-4/3}I_{3,\mathbf{E}} = \frac{2}{3}\begin{bmatrix} 2(1-\alpha^{-3}) & 0 & 0 \\ 0 & 1-\alpha^3 & 0 \\ 0 & 0 & 1-\alpha^3 \end{bmatrix}$$

$$J_{2,\mathbf{E}} = I_3^{-2/3}I_{2,\mathbf{E}} - \frac{2}{3}I_2I_3^{-5/3}I_{3,\mathbf{E}} = \frac{2}{3}\begin{bmatrix} 2(\alpha^{-1}-\alpha^{-4}) & 0 & 0 \\ 0 & -\alpha^2+\alpha^{-1} & 0 \\ 0 & 0 & -\alpha^2+\alpha^{-1} \end{bmatrix}$$

$$J_{3,\mathbf{E}} = \frac{1}{2}I_3^{-1/2}I_{3,\mathbf{E}} = \begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

Thus, the second Piola-Kirchhoff stress becomes

$$\mathbf{S} = A_{10}J_{1,\mathbf{E}} + A_{01}J_{2,\mathbf{E}} + K(J_3 - 1)J_{3,\mathbf{E}}$$

The S_{11} component of the stress becomes

$$\begin{aligned} S_{11} &= \frac{4}{3}\Big[A_{10}(1-\alpha^{-3}) + A_{01}(\alpha^{-1}-\alpha^{-4})\Big] \\ &= \frac{4}{3}(80+20\alpha^{-1})(1-\alpha^{-3}) \end{aligned}$$

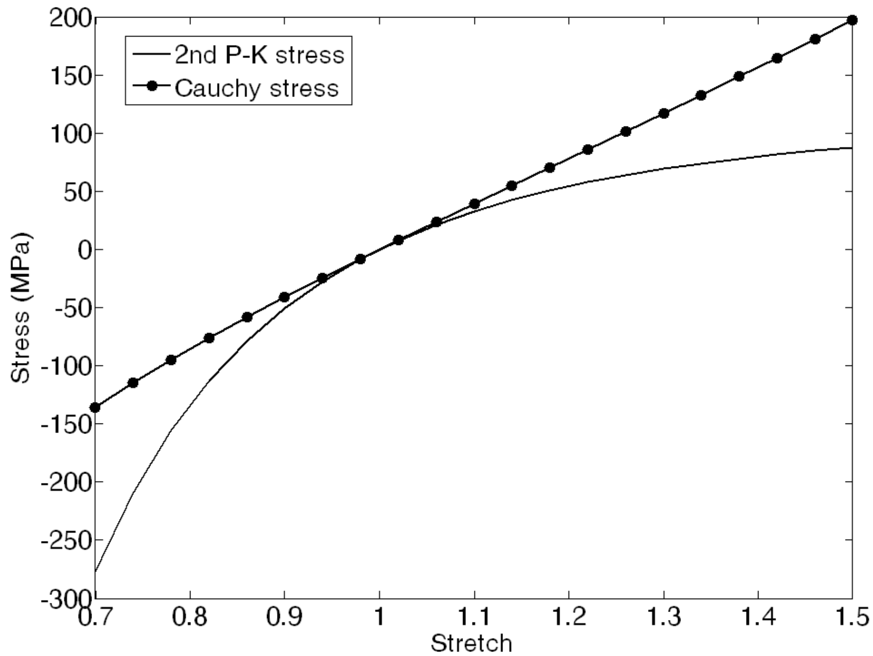
The Cauchy stress can be obtained from the relation

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T$$

The σ_{11} component of the stress becomes

$$\sigma_{11} = \frac{4}{3}(80\alpha^2 + 20\alpha)(1-\alpha^{-3})$$

The following figure shows the two stress components as a function of the principal stretch α. Note that the second Piola-Kirchhoff stress is highly nonlinear, but the Cauchy stress is reasonably linear with respect to the principal stretch. Also note that the two stresses are similar when the deformation is small. However, as deformation increases, the difference also increases.



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2. Solve the same problem using the St. Vernant-Kirchhoff material model. In order to have similar material properties, use $E = 6(A_{10} + A_{01})$ and $\nu = 0.49$ for nearly incompressibility. Compare the two stress components with the stresses from the previous hyperelastic material.

Solution: For the given material properties of $A_{10} = 80$ and $A_{01} = 20$, the Young's modulus becomes $E = 6(A_{10} + A_{01}) = 600 \text{ MPa}$ along with Poisson's ratio of $\nu = 0.49$. Thus, the Lame's constants can be calculated from

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 9{,}865.8 \text{ MPa}$$
$$\mu = \frac{E}{2(1+\nu)} = 201.3 \text{ MPa}$$

For given deformation with incompressibility, the deformation gradient and Green-Lagrange strain tensor can be calculated as

$$\mathbf{F} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \frac{1}{2}\begin{bmatrix} \alpha^2 - 1 & 0 & 0 \\ 0 & \alpha^{-1} - 1 & 0 \\ 0 & 0 & \alpha^{-1} - 1 \end{bmatrix}$$

Since all shear components are zero, we can only consider the normal components as a vector. The second Piola-Kirchhoff stress becomes

$$\mathbf{S} = \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{Bmatrix} = \begin{Bmatrix} \frac{\lambda}{2}(\alpha^2 + 2\alpha^{-1} - 3) + \mu(\alpha^2 - 1) \\ \frac{\lambda}{2}(\alpha^2 + 2\alpha^{-1} - 3) + \mu(\alpha^{-1} - 1) \\ \frac{\lambda}{2}(\alpha^2 + 2\alpha^{-1} - 3) + \mu(\alpha^{-1} - 1) \end{Bmatrix}$$

The S_{11} component of the stress becomes

$$S_{11} = \frac{\lambda}{2}(\alpha^2 + 2\alpha^{-1} - 3) + \mu(\alpha^2 - 1)$$

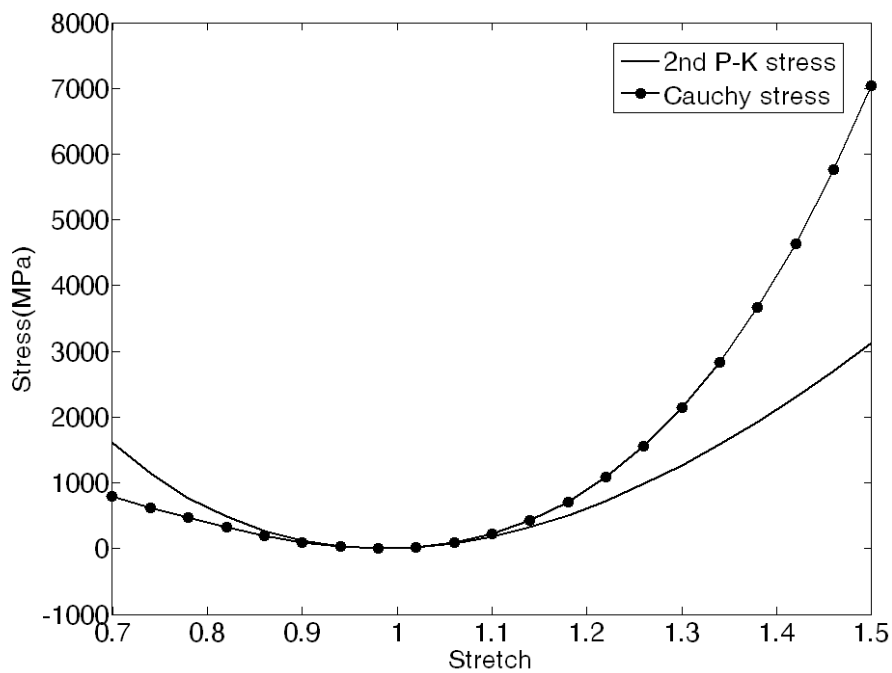
The Cauchy stress can be obtained from the relation

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T$$

The σ_{11} component of the stress becomes

$$\sigma_{11} = \frac{\lambda}{2}(\alpha^4 + 2\alpha - 3\alpha^2) + \mu(\alpha^4 - \alpha^2)$$

The following figure shows the two stress components as a function of the principal stretch α. Note that both stresses are highly nonlinear even if the relation between stress and strain is constant.



3. Consider a simple shear deformation of a square as shown in the figure. The deformed geometry is given as

$$\begin{cases} x_1 = X_1 + \alpha X_2 \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

When the material is incompressible, Mooney-Rivlin hyperelastic material with $A_{10} = 80 \text{ MPa}$ and $A_{01} = 20 \text{ MPa}$, write the expression of S_{12} component of the second Piola-Kirchhoff stress as a function of α. In addition, write the expression of σ_{12} of the Cauchy stress as a function of α. Plot S_{12} and σ_{12} in the range of $\alpha = [0.0\ 1.5]$.

Solution: For given deformation, the deformation gradient and Cauchy-Green deformation tensor can be calculated as

$$\mathbf{F} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & \alpha & 0 \\ \alpha & \alpha^2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The three invariants of the deformation tensor can be obtained as

$$\begin{aligned} I_1 &= \alpha^2 + 3 \\ I_2 &= \alpha^2 + 3 \\ I_3 &= 1 \end{aligned}$$

The reduced invariants become

$$\begin{aligned} J_1 &= I_1 I_3^{-1/3} = \alpha^2 + 3 \\ J_2 &= I_2 I_3^{-2/3} = \alpha^2 + 3 \\ J_3 &= I_3^{-1/2} = 1 \end{aligned}$$

In order to calculate stress, we need to differentiate the reduced invariants with respect to strain

$$I_{1,\mathbf{E}} = 2\mathbf{1} = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_{2,\mathbf{E}} = 2(I_1\mathbf{1} - \mathbf{C}) = \begin{bmatrix} 2\alpha^2 + 4 & -2\alpha & 0 \\ -2\alpha & 4 & 0 \\ 0 & 0 & 2\alpha^2 + 4 \end{bmatrix}$$

$$I_{3,\mathbf{E}} = 2I_3\mathbf{C}^{-1} = \begin{bmatrix} 2\alpha^2 + 2 & -2\alpha & 0 \\ -2\alpha & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The derivatives of the reduced invariants become

$$J_{1,\mathbf{E}} = I_3^{-1/3} I_{1,\mathbf{E}} - \frac{1}{3} I_1 I_3^{-4/3} I_{3,\mathbf{E}} = \frac{2}{3}\begin{bmatrix} -\alpha^4 - 4\alpha^2 & \alpha^3 + 3\alpha & 0 \\ \alpha^3 + 3\alpha & -\alpha^2 & 0 \\ 0 & 0 & -\alpha^2 \end{bmatrix}$$

$$J_{2,\mathbf{E}} = I_3^{-2/3} I_{2,\mathbf{E}} - \frac{2}{3} I_2 I_3^{-5/3} I_{3,\mathbf{E}} = \frac{2}{3}\begin{bmatrix} -2\alpha^4 - 5\alpha^2 & 2\alpha^3 + 3\alpha & 0 \\ 2\alpha^3 + 3\alpha & -2\alpha^2 & 0 \\ 0 & 0 & -2\alpha^2 \end{bmatrix}$$

$$J_{3,\mathbf{E}} = \frac{1}{2} I_3^{-1/2} I_{3,\mathbf{E}} = \begin{bmatrix} \alpha^2 + 1 & -\alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the second Piola-Kirchhoff stress becomes

$$\mathbf{S} = A_{10} J_{1,\mathbf{E}} + A_{01} J_{2,\mathbf{E}} + K(J_3 - 1) J_{3,\mathbf{E}}$$

The S_{12} component of the stress becomes

$$\begin{aligned} S_{12} &= \frac{2}{3}[A_{10}(\alpha^3 + 3\alpha) + A_{01}(2\alpha^3 + 3\alpha)] \\ &= 80\alpha^3 + 200\alpha \end{aligned}$$

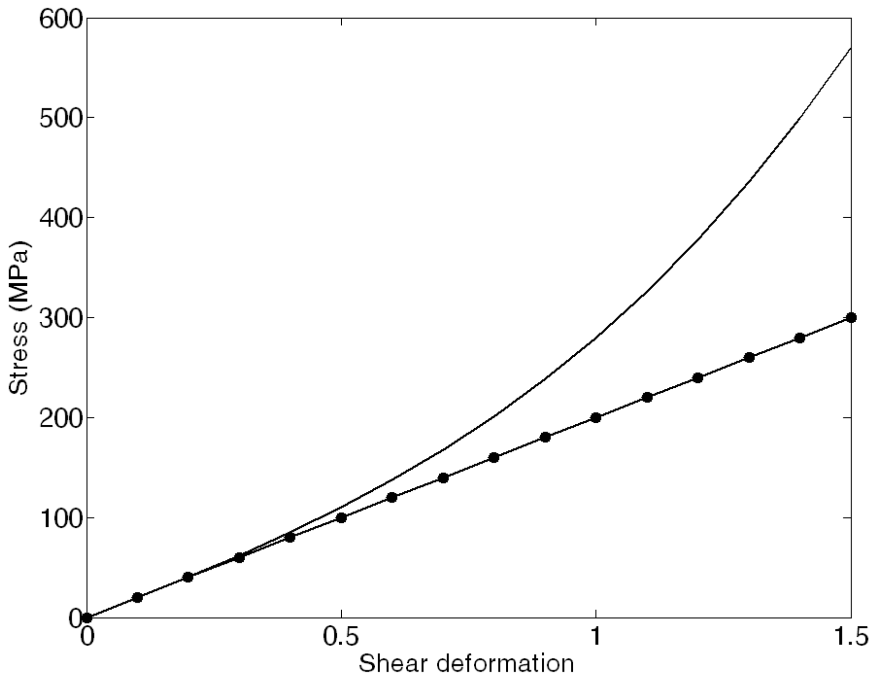
The Cauchy stress can be obtained from the relation

$$\boldsymbol{\sigma} = \frac{1}{J} \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T$$

The σ_{12} component of the stress becomes

$$\sigma_{12} = 2(A_{10} + A_{01})\alpha = 200\alpha$$

Note that the shear stress S_{12} is a cubic function of α, but σ_{12} is a linear function.



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4. Solve the same problem using the St. Vernant-Kirchhoff material model. In order to have similar material properties, use $E = 6(A_{10} + A_{01})$ and $\nu = 0.49$ for nearly incompressibility. Compare the two stress components with the stresses from the previous hyperelastic material.

Solution: For the given material properties of $A_{10} = 80$ and $A_{01} = 20$, the Young's modulus becomes $E = 6(A_{10} + A_{01}) = 600 \text{ MPa}$ along with Poisson's ratio of $\nu = 0.49$. Thus, the Lame's constants can be calculated from

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 9{,}865.8 \text{ MPa}$$
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For given deformation with incompressibility, the deformation gradient and Green-Lagrange strain tensor can be calculated as

$$\mathbf{F} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \frac{1}{2}\begin{bmatrix} 0 & \alpha & 0 \\ \alpha & \alpha^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In the plane strain problem, we can consider only three non-zero stress components. The second Piola-Kirchhoff stress becomes

$$\mathbf{S} = \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{12} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ 2E_{12} \end{Bmatrix} = \begin{Bmatrix} \frac{\lambda}{2}\alpha^2 \\ (\frac{\lambda}{2} + \mu)\alpha^2 \\ \mu\alpha \end{Bmatrix}$$

Thus, S_{12} is a linear function of α. The Cauchy stress can be obtained from the relation

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T$$

The σ_{12} component of the stress becomes

$$\sigma_{12} = \left(\frac{\lambda}{2} + \mu\right)\alpha^3 + \mu\alpha$$

Different from the hyperelastic material, now σ_{12} is a cubic function, while S_{12} is a linear function for the shear deformation. The following figure compares the two stresses as a function of shear deformation.