ESSENTIAL DYNAMICS RELATIONSHIPS

Thee different methods for deriving dynamics equations for systems of rigid bodies connected by joints are described briefly below. The goal is to provide a big picture understanding of these methods without getting lost in the details.

I. NEWTON-EULER METHOD

The Newton-Euler method of deriving dynamics equations involving treating every body as a separate "free body" and solving for ALL reaction forces and torques. Thus, for a 2D problem, you will need to form 3 equations (2 translations and 1 rotational) for every body, while for a 3D problem, you will need to form 6 equations (3 translational and 3 rotational) for each body. If nbodies is the number of bodies, this means that the total number of unknowns will always be 3*nbodies for a 2D problem and 6*nbodies for a 3D problem. These unknowns will be combinations of $\ddot{q}$'s and reaction forces and torques. For example, a 2D problem with a single rigid segment connected to ground by a pin joint will have 3 unknowns that must be solved – one $\ddot{q}$ and two reaction forces (horizontal and vertical). The translational equations come directly from the linear momentum principle, while the rotational equations come directly from the angular momentum principle where the mass center must be chosen as the point about which to sum moments.

II. THE MOTION LAW

The Motion Law (often attributed incorrectly to D'Alambert) replaces the linear and angular momentum principles for a free body *S* (*S* can be a single body or group of bodies) with the related equations

$$\begin{aligned}\mathbf{F}^{S}_{Contact} + \mathbf{F}^{S}_{Dis\tan ce} + \mathbf{F}^{S}_{Inertia} &= \mathbf{0}\\ \mathbf{M}^{S/P}_{Contact} + \mathbf{M}^{S/P}_{Distance} + \mathbf{M}^{S/P}_{Inertia} &= \mathbf{0}\end{aligned}$$

where $\mathbf{F}^{S}_{Contact}$ are all contact forces acting on *S* (e.g., ground contact forces), $\mathbf{F}^{S}_{Dis\tan ce}$ are all distance forces acting on *S* (e.g., gravity forces), and $\mathbf{F}^{S}_{Inertia}$ are all inertia forces acting on *S*. For any individual body *B*, $\mathbf{F}^{B}_{Inertia}$ can be easily calculate as

$$\mathbf{F}^{B}_{Inertia} = -m_B\,{}^{N}\mathbf{a}^{B}$$

which is equivalent to $-{}^{N}d\,{}^{N}\mathbf{L}^{B}/dt$ for body *B*.

As indicated above, these contact, distance, and inertia forces also have moment counterparts. However, unlike the angular momentum principle, the moments may be taken about ANY point *P*. This is the primary advantage of the Motion Law over the Newton-Euler method. This advantage eliminates the need to solve large systems of equations for large numbers of unwanted reaction forces and torques.

Furthermore, it makes the formulation and solution of dynamics problems identical conceptually to the solution of statics problems. The contact moment $\mathbf{M}^{S/P}_{Contact}$ is simply the moment about point *P* of all contact forces acting on free body *S*. Similarly, the distance moment $\mathbf{M}^{S/P}_{Distance}$ is the moment about point *P* of all distance forces acting on *S*. The inertia moment $\mathbf{M}^{S/P}_{Inertia}$ is the quantity that is different, as it is composed of two contributions: the moment of all inertia forces about point *P*, plus a new quantity called an inertia torque. For a rigid body *B*, the inertia torque is defined as

$$\mathbf{T}^{B/B*} = -{}^{N}\boldsymbol{\alpha}^{B} \cdot \mathbf{I}^{B/B*} - {}^{N}\boldsymbol{\omega}^{B} \times \mathbf{I}^{B/B*} \cdot {}^{N}\boldsymbol{\omega}^{B}$$

which is equivalent to $-{}^{N}d\,{}^{N}\mathbf{H}^{B/B*}/dt$ for body *B* based on the Million Dollar Formula. Thus, the inertia moment $\mathbf{M}^{B/P}_{Inertia}$ for body *B* about point *P* becomes

$$\mathbf{M}^{B/P}_{Inertia} = \mathbf{p}^{PB*} \times \mathbf{F}^{B}_{Inertia} + \mathbf{T}^{B/B*}$$

Note that if point *P* is taken as $B*$, the mass center of body *B*, then the cross product above goes to zero, and the inertia moment becomes identical to the inertia torque.

For a more thorough discussion of the Moton Law, consult Kane and Levinson (1997 and 1998).

III. KANE'S METHOD

Kane's method can be viewed as an automated version of the Motion Law, where no analyst decisions are required to produce a minimal set of dynamics equations in which unwanted reaction forces and torques do not appear. The key concept underlying Kane's method is that of partial velocities, which are described in more detail below.

A. Partial Velocities . . .

Generalized coordinates q_r - time-varying translations and rotations selected to define the position of all points and the orientation of all rigid bodies.

$$q_r \quad (r = 1, \ldots, n) \tag{1}$$

where *n* is the number of degrees of freedom.

Generalized speeds u_r - time-varying linear functions of the $\dot{q}_r$'s selected so as to simplify expressions for velocities of points and angular velocities of rigid bodies.

$$u_r \overset{\Delta}{=} \sum_{s=1}^{n} Y_{rs}\dot{q}_s + Z_r \quad (r = 1, \ldots, n) \tag{2}$$

where Y_{rs} and Z_r are functions of $q_1, \ldots, q_n$ and the time *t*. Note that while $u_r \overset{\Delta}{=} \dot{q}_r$ is the simplest and most obvious definition, a more advantageous, though more complex, definition may also exist (see, for example, Mitiguy and Kane, 1996). For any definition, Eqs. (2) must yield unique solutions for $\dot{q}_1, \dots, \dot{q}_n$ as a function of $u_1, \dots, u_n$.

Example: Consider the constrained velocity ${}^N\mathbf{v}^P$ of a fictitious particle *P* in a Newtonian reference frame *N*, where *P* is part of a larger system requiring three generalized coordinates:

$${}^N\mathbf{v}^P = (\dot{q}_1 \cos q_3 + \dot{q}_2 \sin q_3)\mathbf{n}_1 + (-\dot{q}_1 \sin q_3 + \dot{q}_2 \cos q_3)\mathbf{n}_2 - q_1\mathbf{n}_3$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ form a right-handed set of mutually perpendicular unit vectors fixed in *N*. If we define

$$u_1 \overset{\Delta}{=} \dot{q}_1 \cos q_3 + \dot{q}_2 \sin q_3 \qquad u_2 \overset{\Delta}{=} -\dot{q}_1 \sin q_3 + \dot{q}_2 \cos q_3 \qquad u_3 \overset{\Delta}{=} \dot{q}_3$$

then we can rewrite ${}^N\mathbf{v}^P$ as

$${}^N\mathbf{v}^P = u_1\mathbf{n}_1 + u_2\mathbf{n}_2 - q_1\mathbf{n}_3$$

Partial angular velocities $\boldsymbol{\omega}_r$ *and partial velocities* $\mathbf{v}_r$ - time-varying linear functions of the u_r's determined by *inspection* and which greatly facilitate the formulation of equations of motion.

$$\begin{aligned} \boldsymbol{\omega} &= \sum_{r=1}^{n} \boldsymbol{\omega}_r u_r + \boldsymbol{\omega}_t \\ \mathbf{v} &= \sum_{r=1}^{n} \mathbf{v}_r u_r + \mathbf{v}_t \end{aligned} \tag{3}$$

where $\boldsymbol{\omega}$ is the angular velocity of a rigid body, $\mathbf{v}$ is the velocity of a point, and $\boldsymbol{\omega}_r$, $\mathbf{v}_r$, $\boldsymbol{\omega}_t$, and $\mathbf{v}_t$ are functions of $q_1, \dots, q_n$ and *t*. In principle, partial angular velocities need only be formed for those rigid bodies subjected to applied torques or possessing inertia, while partial velocities need only be formed for those points subjected to applied forces or possessing mass.

Example: By rewriting the above expression for ${}^N\mathbf{v}^P$ as

$${}^N\mathbf{v}^P = (\mathbf{n}_1)u_1 + (\mathbf{n}_2)u_2 + (0)u_3 + (-q_1\mathbf{n}_3)$$

the three partial velocities associated with particle *P* are found to be

$${}^N\mathbf{v}_1^P = \mathbf{n}_1 \qquad {}^N\mathbf{v}_2^P = \mathbf{n}_2 \qquad {}^N\mathbf{v}_3^P = 0$$

while ${}^N\mathbf{v}_t^P = -q_1\mathbf{n}_3$.

B. And Their Usefulness

Generalized active forces F_r - quantities formed by taking dot (i.e., scalar) products of partial velocities and active (i.e., applied) forces and dot products of partial angular velocities and active torques. For each point P_i subjected to an applied force,

$$(F_r)_{P_i} = \mathbf{v}_r^{P_i} \cdot \mathbf{R}_{P_i} \quad (r = 1, \dots, n) \tag{4}$$

where $\mathbf{v}_r^{P_i}$ is the r^{th} partial velocity of P_i and $\mathbf{R}_{P_i}$ is the resultant of all contact and distance forces acting on P_i. Similarly, for each rigid body B_j subjected to an applied torque,

$$(F_r)_{B_j} = \boldsymbol{\omega}_r^{B_j} \cdot \mathbf{T}_{B_j} \quad (r = 1, \dots, n) \tag{5}$$

where $\boldsymbol{\omega}_r^{B_j}$ is the r^{th} partial angular velocity of B_j and $\mathbf{T}_{B_j}$ is the resultant of all couples acting on B_j. The r^{th} generalized active force F_r can then be determined by summing the results over all points P_i and all rigid bodies B_j:

$$F_r = \sum_{i=1}^{\kappa} (F_r)_{P_i} + \sum_{j=1}^{\lambda} (F_r)_{B_j} \quad (r = 1, \dots, n) \tag{6}$$

where κ is the number of points subjected to applied forces and λ is the number of rigid bodies subjected to applied torques.

Generalized inertia forces F_r* - quantities formed by taking dot products of partial velocities and inertia forces and dot products of partial angular velocities and inertia torques. For each point P_i possessing mass,

$$(F_r*)_{P_i} = \mathbf{v}_r^{P_i} \cdot \mathbf{R}_{P_i}* \quad (r = 1, \dots, n) \tag{7}$$

where $\mathbf{v}_r^{P_i}$ is the r^{th} partial velocity of P_i and $\mathbf{R}_{P_i}*$ is the inertia force for P_i, defined as

$$\mathbf{R}_{P_i}* \triangleq -m_{P_i}\mathbf{a}^{P_i} \tag{8}$$

where m_{P_i} is the mass of P_i and $\mathbf{a}^{P_i}$ is the acceleration of P_i. Similarly, for each rigid body B_j possessing inertia,

$$(F_r*)_{B_j} = \boldsymbol{\omega}_r^{B_j} \cdot \mathbf{T}_{B_j}* \quad (r = 1, \dots, n) \tag{9}$$

where $\boldsymbol{\omega}_r^{B_j}$ is the r^{th} partial angular velocity of B_j and $\mathbf{T}_{B_j}*$ is the inertia torque for B_j, defined as

$$\mathbf{T}_{B_j}* \triangleq -\boldsymbol{\alpha}^{B_j} \cdot \mathbf{I}^{B_j/B_j*} - \boldsymbol{\omega}^{B_j} \times \mathbf{I}^{B_j/B_j*} \cdot \boldsymbol{\omega}^{B_j} \tag{10}$$

where $\mathbf{I}^{B_j/B_j*}$ is the inertia dyadic of B_j about its mass center B_j*, $\boldsymbol{\omega}^{B_j}$ is the angular velocity of B_j, and $\boldsymbol{\alpha}^{B_j}$ is the angular acceleration of B_j. Note that a dyadic is an expression of the form

$$\begin{aligned} I &= \sum_{j=1}^{3}\sum_{i=1}^{3} I_{jk}\boldsymbol{b}_j\boldsymbol{b}_k \\ &= \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{bmatrix} \end{aligned} \tag{11}$$

The r^{th} generalized inertia force F_r * can then be determined by summing the results over all points P_i and all rigid bodies B_j:

$$F_r^* = \sum_{i=1}^{\mu} (F_r^*)_{P_i} + \sum_{j=1}^{\nu} (F_r^*)_{B_j} \tag{12}$$

where μ is the number of points possessing mass and ν is the number of rigid bodies possessing inertia.

Equations of motion $F_r + F_r^* = 0$ - once all generalized active forces and generalized inertia forces are known, the equations of motion can be formulated by simply adding the results:

$$F_r + F_r^* = 0 \quad (r = 1, \ldots, n) \tag{13}$$

One arrives at Eq. (13) by following a very systematic process which does not require high level mathematics, calculation of unwanted interaction forces, or use of virtual work principles. Note that statics problems can be solved by considering $F_r = 0$.

IV. SOLUTION PROCESS

1) Define reference frames, unit vectors fixed in each reference frame, and unknown translations and rotations corresponding to the degrees of freedom.
2) Draw free body diagrams, realizing that groups of individual bodies can be treated as a "free body."
3) Apply the Newton-Euler, Motion Law, or Kane method, choosing directions in which to sum forces and points about which to sum moments in such as way as to minimize the need to calculate unwanted reaction forces and moments. Create a table listing which body or group of bodies will be treated as the "free body," which equation is to be used (e.g., $\mathbf{F}$ to represent sum of forces and $\mathbf{M}^P$ to represent the moment about some point *P*), in which direction a dot product will be taken, and finally what the resulting unknowns will be (i.e., second time derivatives of the unknown translations and rotations and possibly reaction forces and torques).
4) Calculate important kinematic quantities (i.e., the acceleration of every point with mass, the angular velocity and acceleration of every body with inertia) with respect to the inertial reference frame and express them in terms of time derivatives of the unknown translations and rotations.
5) Solve the resulting linear system of equations for the second time derivatives of the unknown translations and rotations and for the unknown reaction forces and torques (if present).
6) Integrate the second time derivative equations numerically to determine the unknown translations and rotations as a function of time.

References

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