SOLUTIONS OF THE BRAHMAGUPTA-PELL EQUATION

One of the most important non-linear Diophantine equations is-

\[ y^2 = 1 + Nx^2, \quad \text{where } N \text{ is a NonSquare Integer} \]

One seeks solutions in the form of integer pairs \((x_n,y_n)\). The equation, which I will refer to as the BP equation, was first studied by the Indian mathematician Brahmagupta (598-670AD), who is better known as the originator of the area of a quadrilateral inscribed in a circle. John Pell, an English mathematician of the sixteen hundreds, got his name attached to the equation through an attribution error made by Euler. The first thing one notices about the equation is that one can write it as the product-

\[
1 = (y_1^2 - Nx_1^2)(y_2^2 - Nx_2^2) = (y_1y_2)^2 - N[(y_1x_2)^2 + (x_1y_2)^2] + N^2(x_1x_2)^2
\]

Thus if \((x_1,y_1)\) and \((x_2,y_2)\) are integer solutions, then so is \((x_3,y_3)\) where-

\[
x_3 = y_1x_2 + x_1y_2, \quad \text{and} \quad y_3 = y_1y_2 + Nx_1x_2
\]

Here \((x_1,y_1)\) is referred to as the fundamental solution pair. More generally we have that the solution pair \((x_n,y_n)\) relates to the next highest solution pair \((x_{n+1},y_{n+1})\) via the relation-

\[
x_{n+1} = y_1x_n + x_1y_n \quad \text{and} \quad y_{n+1} = y_1y_n + Nx_1x_n
\]

Therefore, once the value of the fundamental pair has been found, all higher pairs follow automatically from this last relation by rewriting things as -

\[
y_{n+1} + x_{n+1}\sqrt{N} = [y_1 + x_1\sqrt{N}][y_n + x_n\sqrt{N}]
\]

or-

\[
y_n + x_n\sqrt{N} = [y_1 + x_1\sqrt{N}]^a
\]

We confirm this last result by noting that the first three solution pairs when \(N=2\) are (2,3), (12,17), and (70,99). Thus-

\[
(3 + 2\sqrt{2})^2 = 17 + 12\sqrt{2} \quad \text{and} \quad (3 + 2\sqrt{2})^3 = 99 + 70\sqrt{2}
\]
The 4th solution pair will be given by-

$$(99 + 70\sqrt{2})(3 + 2\sqrt{2}) = 577 + 408\sqrt{2}$$

That is, $(x_4, y_4) = (408, 577)$.

In evaluating $x_n$ and $y_n$ for various fixed values of integer $N$, one notices that the ratios of $x_{n+1}/x_n$ and $y_{n+1}/y_n$ approach the same constant value. We can calculate what these ratios are by noting that-

$$\frac{x_{n+1}}{x_n} = y_1 + \frac{x_1 y_n}{x_n} \quad \text{and} \quad \frac{y_{n+1}}{y_n} = y_1 + \frac{x_1 N x_n}{y_n}$$

On equating these to each other, we find-

$$\frac{x_{n+1}}{x_n} = y_1 + x_1 \sqrt{N} = \frac{y_{n+1}}{y_n}$$

Thus, for $N=2$ the solution pair ratios go as $3 + 2\sqrt{2}$ and for $N=10$ where $(x_1, y_1) = (6, 19)$ the ratios equal $19 + 6\sqrt{10} = 37.97366597\ldots$. Values of the first (fundamental) and second solution pairs for $N=2$ through 12 follow-

<table>
<thead>
<tr>
<th>$N$</th>
<th>$(x_1, y_1)$</th>
<th>$(x_2, y_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2, 3</td>
<td>12, 17</td>
</tr>
<tr>
<td>3</td>
<td>1, 2</td>
<td>4, 7</td>
</tr>
<tr>
<td>4</td>
<td>only 0, 1</td>
<td>none</td>
</tr>
<tr>
<td>5</td>
<td>4, 9</td>
<td>72, 161</td>
</tr>
<tr>
<td>6</td>
<td>2, 5</td>
<td>20, 49</td>
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<tr>
<td>7</td>
<td>3, 8</td>
<td>48, 127</td>
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<tr>
<td>8</td>
<td>1, 3</td>
<td>6, 17</td>
</tr>
<tr>
<td>9</td>
<td>only 0, 1</td>
<td>none</td>
</tr>
<tr>
<td>10</td>
<td>6, 19</td>
<td>228, 721</td>
</tr>
<tr>
<td>11</td>
<td>3, 10</td>
<td>60, 199</td>
</tr>
<tr>
<td>12</td>
<td>2, 7</td>
<td>28, 97</td>
</tr>
</tbody>
</table>
Another interesting fact about the BP equation, as first noted by Euler, is that it may be expanded as a continued fraction as follows:

\[ y = x \sqrt{N} + \frac{1}{y + x \sqrt{N}} = x \sqrt{N} + \frac{1}{2x \sqrt{N} + \frac{1}{2x \sqrt{N} + \frac{1}{2x \sqrt{N} + \ddots}}} \]

Multiplying both sides by \( \sqrt{N} \) yields the root equation:

\[ \sqrt{N} = xN + \frac{1}{y} \left\{ \frac{1}{2x + \frac{1}{2xN + \frac{1}{2x + \ddots}}} \right\} \approx \frac{[xN + \frac{1}{2x}]}{y} \text{ when } x, y >> 1 \]

Thus taking the fifth BK pair (408,577) for \( N=2 \), we have:

\[ \sqrt{2} \approx \frac{[408(2) + \frac{1}{2(408)}]}{577} = 1.41421356237... \]

which is accurate to nine places. Using even larger solution pairs of the BK equation will further improve this already impressive accuracy.

Alternatively one can let \( z = x \sqrt{N} = \sqrt{y^2 - 1} \), to find:

\[ \left[ \frac{1}{\sqrt{z^2 + 1} - z} \right] = \frac{1}{2z + \frac{1}{2z + \frac{1}{2z + \ddots}}} \]
This should work for any \( z \) and not really depend on a BK equation solution. It leads to the much slower converging continued fraction expansion:

\[
\sqrt{2} - 1 = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ddots}}}
\]

An even better root formula using the BK equation is gotten by the following Taylor series expansion:

\[
y = \sqrt{1 + Nx^2} = x\sqrt{N} \left\{ 1 + \frac{1}{1!2(Nx^2)} - \frac{1}{2!4(Nx^2)^2} + \frac{1\cdot3}{3!8(Nx^2)^3} - \right\}
\]

Multiplying by both sides by \( \sqrt{N} \) and dividing by \( y \), yields the root equation:

\[
\sqrt{N} = \frac{xN}{y} \left\{ 1 + \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{n!(1+n)!2^{2n+1}(Nx^2)^{n+1}} \right\}
\]

This series converges extremely rapidly when \( x \) becomes large. Consider the tenth solution pair of the BK equation at \( N=2 \). It reads:

\[
(x_{10}, y_{10}) = (15994428, 22619537)
\]

and is generated by:

\[
(3 + 2\sqrt{2})^{10} = 22619537 + 15994428\sqrt{2}
\]

Programming the series and taking things only the four terms, we obtain the 83 digit accurate result:

\[
\text{sqrt}(2) = 1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388
\]
An not quite as accurate, but still an excellent, approximation for a root is given by just the first term:

\[ \sqrt{N} \approx \frac{xN}{y} \quad \text{provided} \quad x \gg 1 \]

Using the known result \((x_1, y_1) = (226153980, 1766319049)\) at \(N=61\), we have that:

\[
\sqrt{61} \approx \frac{61(226153980)}{1766319049} = 7.81024967590665439.. 
\]

This result is accurate to 18 places.

A major drawback of finding accurate approximations to roots of numbers by the above method is that it is very time consuming to find the fundamental solution \((x_1, y_1)\) to the BP equation when \(N\) becomes large, especially when \(N\) is a prime number. For example, in searching for this pair by the brute force method of looking at the function \(\sqrt{1+619n^2}\), we could not find any integer solutions for \(n\) from 1 through 4000. So all we know is that the fundamental solution lies above \(x_1=4000\) and \(y_1=4000\sqrt{619}=99519\) for the prime \(N=619\). I leave it for the readers to find the first solution pair for:

\[ y^2 = 1 + 619x^2 \]

The task is a bit easier when \(N\) is non-prime such as \(621=27\times23\). Here the fundamental for to \(N=27\) is \((5, 26)\) and for \(N=23\) is \((5, 24)\) so that the root to 621 is easy to approximate by taking the product of smaller integer root approximations and then multiplying things together. Using the \(n=5\) solution pairs, one finds (on using \(\sqrt{N} \approx Nx_5/y_5\)) a square root of 621 approximation good to 14 decimal places.

September 2011