DERIVATION OF THE STOKES DRAG FORMULA

In an important paper of 1851, G. Stokes first derived the basic formula for the drag on a sphere of radius r=a moving with speed U_o through a viscous fluid of density ρ and viscosity coefficient μ . The formula reads-

$$\mathbf{F=6\,\pi\,\mu\,a\,U_o}$$

It applies strictly only in the creeping flow regime where the Reynolds number $Re = \rho a U_o/\mu$ is less than unity and is thus mainly concerned with spheres of very small diameter when the surrounding fluid is either a gas or a liquid. The formula has found wide application ranging from determining the basic charge of an electron to predicting the settling velocity of suspended sediments. In the bio-fluids area it is encountered when studying the settling rates of blood cells in centrifugal fields and in determining the rate at which small contaminants are deposited in the lungs.

Most introductory texts (and also many more advanced fluid books) do not give a full derivation of Stokes's Drag Law mainly because of the mathematical complexity involved. We help remedy this situation here by presenting a full derivation.

We begin by introducing a spherical coordinate system with the sphere of radius r=a placed at its center. The viscous and incompressible flow about this sphere will , by symmetry considerations, have only a radial and a polar angle dependent velocity component. The fact that the divergence of this velocity field is zero allows one to introduce the stream function Ψ(r ,θ) defined by-

$$V_r = [1/(r^2 \sin\theta)]\partial\Psi/\partial\theta \quad \text{and} \quad V_\theta = -[1/(r \sin\theta)]\partial\Psi/\partial r$$

Next one looks at the two momentum equations for a steady creeping flow. Using the fact that $\nabla^2\mathbf{V}$ can be replaced by -curl (curl **V**) for an incompressible fluid of zero divergence, these equations can, after a little manipulation , be re-written as-

$$\partial P/\partial r = [\mu/(r^2 \sin\theta)]\partial(Q\Psi)/\partial\theta \quad \text{and} \quad \partial P/\partial\theta = -[\mu/(r\sin\theta)]\partial(Q\Psi)/\partial r$$

where Q a differential operator defined as-

$$Q = \partial^2/\partial r^2 + (\sin\theta/r^2)\,\partial/\partial\theta[(1/\sin\theta)\partial/\partial\theta]$$

Eliminating p between these momentum equations , one finds the fourth order PDE

$$Q^2\Psi = 0$$

Making the substitution -

$$\Psi = f(r)\sin^2\theta$$

allows one to reduce this equation to a 4th order ODE of the Euler type , namely-

$$r^4d^4f/dr^4-4r^2d^2f/dr^2+8rdf/dr-8f = 0$$

This equation is readily solved via the substitution $f=r^n$ and leads to the solution -

$$f = A/r+Br+Cr^2+Dr^4$$

We can evaluate the four constants A, B, C, and D from the fact that both velocity components vanish at the sphere surface at r=a and that V_r goes as $U_o \cos\theta$ as r gets large. This leads to the Stokes streamfunction-

$$\Psi = U_o\ [a^3/4r - 3ar/4 + r^2/2]\sin^2\theta$$

From this last result , and use of the definitions for the velocity components and radial pressure gradient given above, one also finds that-

$$V_r = U_o[a^3/2r^3-3a/2r+1]\cos\theta$$

$$V_\theta = U_o[a^3/4r^3+3a/4r-1]\sin\theta$$

and

$$P = -\ [(3a\mu U_o)/(2r^2)]\cos\theta$$

The drag force can now be calculated by integrating the shear stress and normal stress over the entire surface of the sphere. In this calculation p is the normal stress and the shear stress is-

$$\tau_{r\,\theta} = -\mu[r\partial/\partial r(V_\theta/r)+(1/r)(\partial V_r/\partial\theta)]$$

Carrying out the integration from $\theta=0$ to π for the pressure induced drag, one has a net force in the z direction of-

$$F_{pressure} = 2\pi a^2 \int \sin\theta \cos\theta \; p \; d\theta = 2\pi a \mu U_o$$

Next the force in the z direction due to viscous shear stress yields-

$$F_{shear} = 2\pi a^2 \int \sin^2\theta \; \tau_{r\,\theta} \; d\theta = 4\pi a \mu U_o$$

Adding things together we arrive at the famous Stokes Drag Law first derived by him over 150 years ago! Note that in this problem two thirds of the force on the sphere is due to viscous shear and only one third due to pressure drag.

When dealing with small spheres dropping with constant speed in a gravity field one must equate the Stokes drag to the effective downward force equal to the sphere weight minus the buoyancy force. This yields -

$$U_o = 2g\,(a^2/9\,\mu)[\rho_{sphere} - \rho_{fluid}]$$

From this result one can infer that a 20 micron diameter iron sphere dropping in quiescent water will have a terminal downward velocity of just 1.5 mm/sec. The corresponding Reynolds number will be Re=0.015 and so lies well within the creeping flow regime. Since the terminal velocity is proportional to the square of the sphere diameter one could use such a dropping procedure to segregate out different size spheres from a mixed batch.