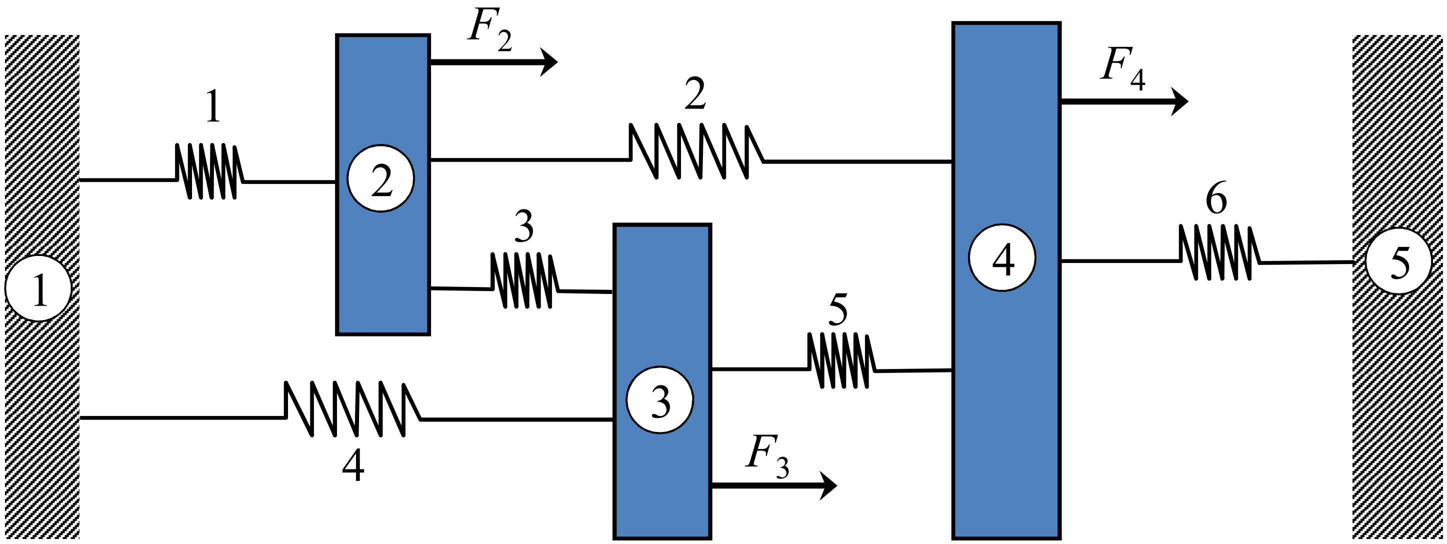
UF UNIVERSITY of FLORIDA
The Foundation for The Gator Nation

Ch 1 Spring, Uniaxial Bar and Truss Element

1-D System of Springs

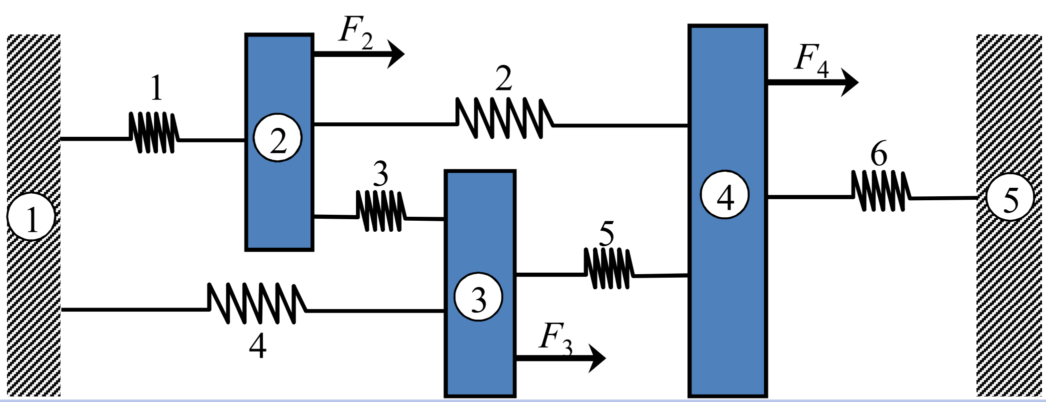


- Bodies move only in horizontal direction
- External forces, F_2, F_3, and F_4, are applied
- No need to discretize the system (it is already discretized!)
- Rigid body (including walls) ⇒ **NODE**
- Spring ⇒ **ELEMENT**

Connectivity Table

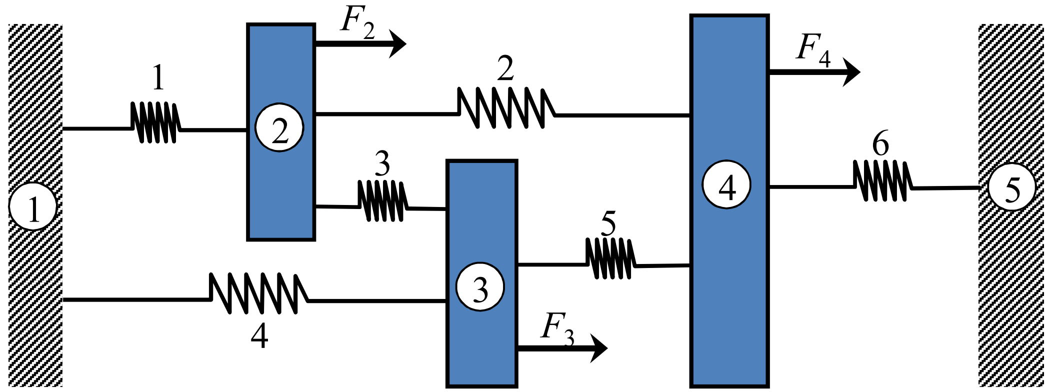
- Mesh: system of connected elements
- Connectivity: Local node 1 (i) → Local node 2 (j)

Element	LN1 (i)	LN2 (j)
1	1	2
2	2	4
3	2	3
4	1	3
5	3	4
6	4	5



System of Springs *cont.*

- Element equation and assembly



$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_4^{(2)} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ 0 \\ f_4^{(2)} \\ 0 \end{Bmatrix}$$

System of Springs *cont.*

$$\begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}=\begin{Bmatrix} f_2^{(3)} \\ f_3^{(3)} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ 0 & -k_3 & k_3 & 0 & 0 \\ 0 & -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}=\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)}+f_2^{(2)}+f_2^{(3)} \\ f_3^{(3)} \\ f_4^{(2)} \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix}=\begin{Bmatrix} f_1^{(4)} \\ f_3^{(4)} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_1+k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3+k_4 & 0 & 0 \\ 0 & -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}=\begin{Bmatrix} f_1^{(1)}+f_1^{(4)} \\ f_2^{(1)}+f_2^{(2)}+f_2^{(3)} \\ f_3^{(3)}+f_3^{(4)} \\ f_4^{(2)} \\ 0 \end{Bmatrix}$$

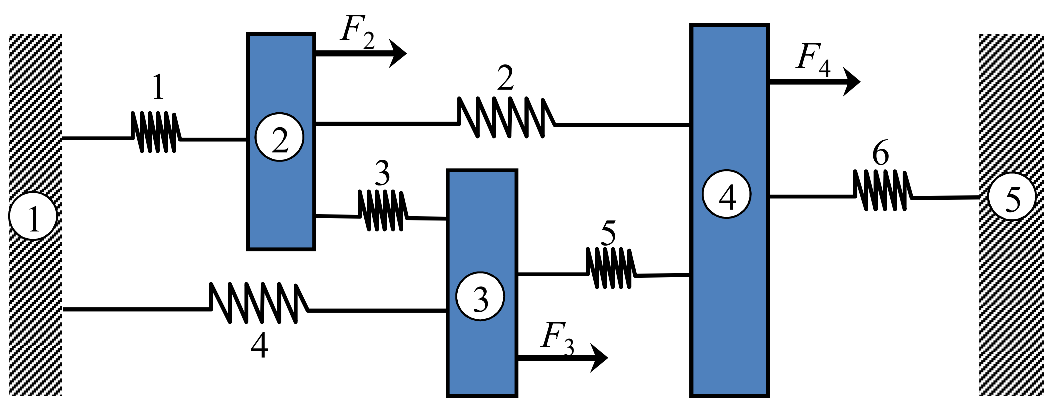
$$\begin{bmatrix} k_5 & -k_5 \\ -k_5 & k_5 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}=\begin{Bmatrix} f_3^{(5)} \\ f_4^{(5)} \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} k_1+k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3+k_4+k_5 & -k_5 & 0 \\ 0 & -k_2 & -k_5 & k_2+k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}=\begin{Bmatrix} f_1^{(1)}+f_1^{(4)} \\ f_2^{(1)}+f_2^{(2)}+f_2^{(3)} \\ f_3^{(3)}+f_3^{(4)}+f_3^{(5)} \\ f_4^{(2)}+f_4^{(5)} \\ 0 \end{Bmatrix}$$

System of Springs *cont.*

$$\begin{bmatrix} k_6 & -k_6 \\ -k_6 & k_6 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_4^{(6)} \\ f_5^{(6)} \end{Bmatrix} \Rightarrow$$

$$\begin{bmatrix} k_1 + k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3 + k_5 + k_4 & -k_5 & 0 \\ 0 & -k_2 & -k_5 & k_2 + k_5 + k_6 & -k_6 \\ 0 & 0 & 0 & -k_6 & +k_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} + f_1^{(4)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} + f_3^{(4)} + f_3^{(5)} \\ f_4^{(2)} + f_4^{(5)} + f_4^{(6)} \\ f_5^{(6)} \end{Bmatrix}$$

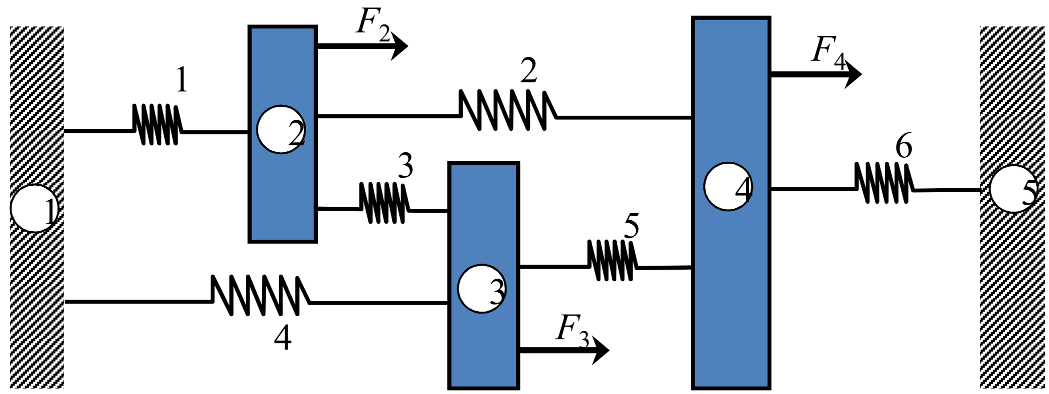


System of Springs *cont.*

- Relation b/w element forces and external force
- Force equilibrium

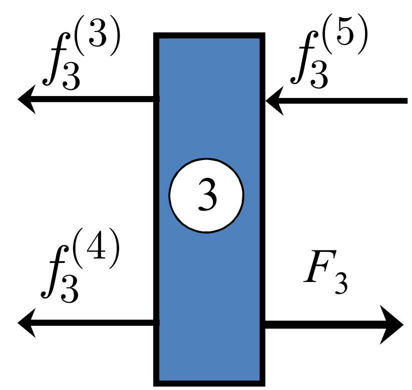
$$F_i - \sum_{e=1}^{i_e} f_i^{(e)} = 0$$

$$F_i = \sum_{e=1}^{i_e} f_i^{(e)}, \quad i = 1,\ldots ND$$



- At node 3

$$F_3 - f_3^{(3)} - f_3^{(4)} - f_3^{(5)} = 0$$



- At each node, the summation of **element forces** is equal to the **applied, external force**

$$\begin{Bmatrix} f_1^{(1)} + f_1^{(4)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} + f_3^{(4)} + f_3^{(5)} \\ f_4^{(2)} + f_4^{(5)} + f_4^{(6)} \\ f_5^{(6)} \end{Bmatrix} = \begin{Bmatrix} R_1 \\ F_2 \\ F_3 \\ F_4 \\ R_5 \end{Bmatrix}$$

System of Springs *cont.*

- Assembled System of Matrix Equation:

$$\begin{bmatrix} k_1+k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3+k_5+k_4 & -k_5 & 0 \\ 0 & -k_2 & -k_5 & k_2+k_5+k_6 & -k_6 \\ 0 & 0 & 0 & -k_6 & +k_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ F_2 \\ F_3 \\ F_4 \\ R_5 \end{Bmatrix}$$

$$[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\}$$

- $[\mathbf{K}_s]$ is square, symmetric, singular and positive semi-definite.
- When displacement is known, force is unknown

$u_1 = u_5 = 0 \Longrightarrow$ R_1 and R_5 are unknown reaction forces

System of Springs *cont.*

- Imposing Boundary Conditions
  - Ignore the equations for which the RHS forces are unknown and strike out the corresponding rows in [$\mathbf{K}_s$].
  - Eliminate the columns in [$\mathbf{K}_s$] that multiply into zero values of displacements of the boundary nodes.

$$\begin{bmatrix} k_1+k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3+k_5+k_4 & -k_5 & 0 \\ 0 & -k_2 & -k_5 & k_2+k_5+k_6 & -k_6 \\ 0 & 0 & 0 & -k_6 & +k_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ F_2 \\ F_3 \\ F_4 \\ R_5 \end{Bmatrix}$$

System of Springs *cont.*

- Global Matrix Equation

$$\begin{bmatrix} k_1 + k_2 + k_3 & -k_3 & -k_2 \\ -k_3 & k_3 + k_4 + k_5 & -k_5 \\ -k_2 & -k_5 & k_2 + k_5 + k_6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\}$$

- Global Stiffness Matrix [**K**]
  - square, symmetric and positive definite and hence non-singular
- Solution

$$\{\mathbf{Q}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\}$$

- Once nodal displacements are obtained, spring forces can be calculated from

$$P^{(e)} = k^{(e)}\, \Delta^{(e)} = k^{(e)}(u_j - u_i)$$

Statically Indeterminate System

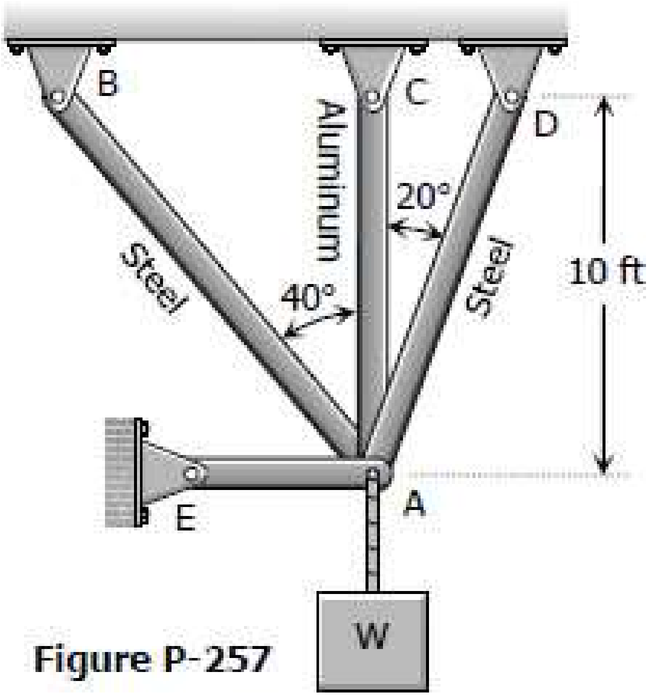
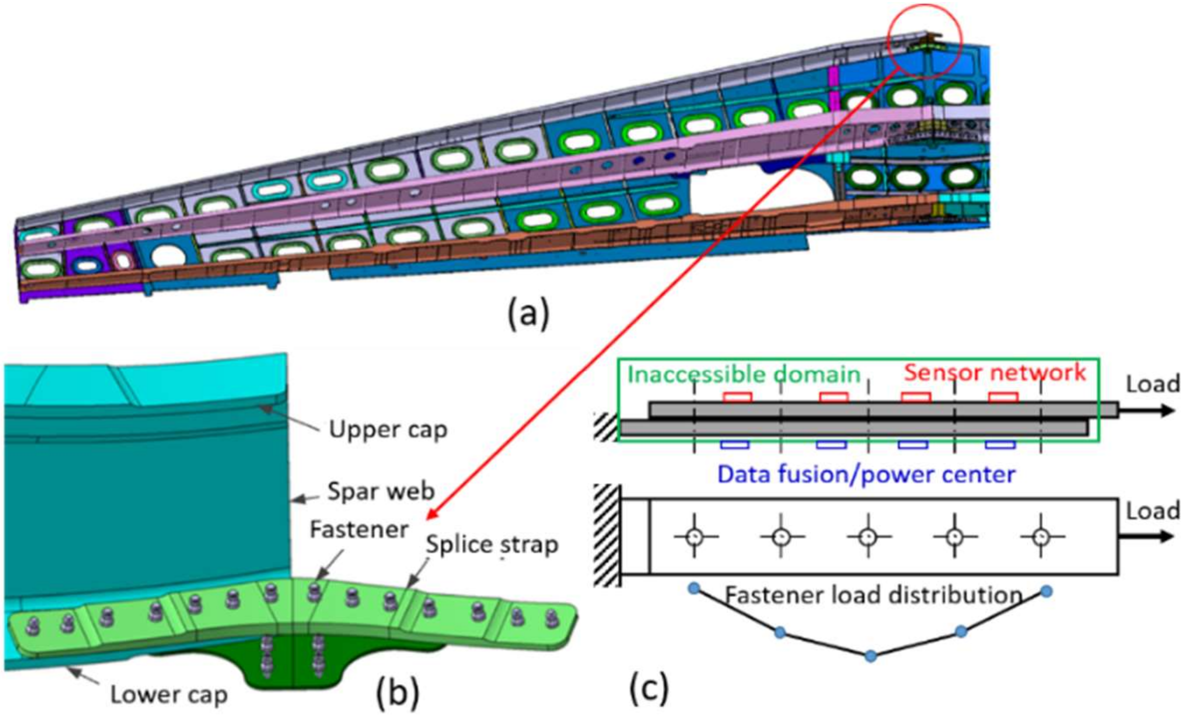
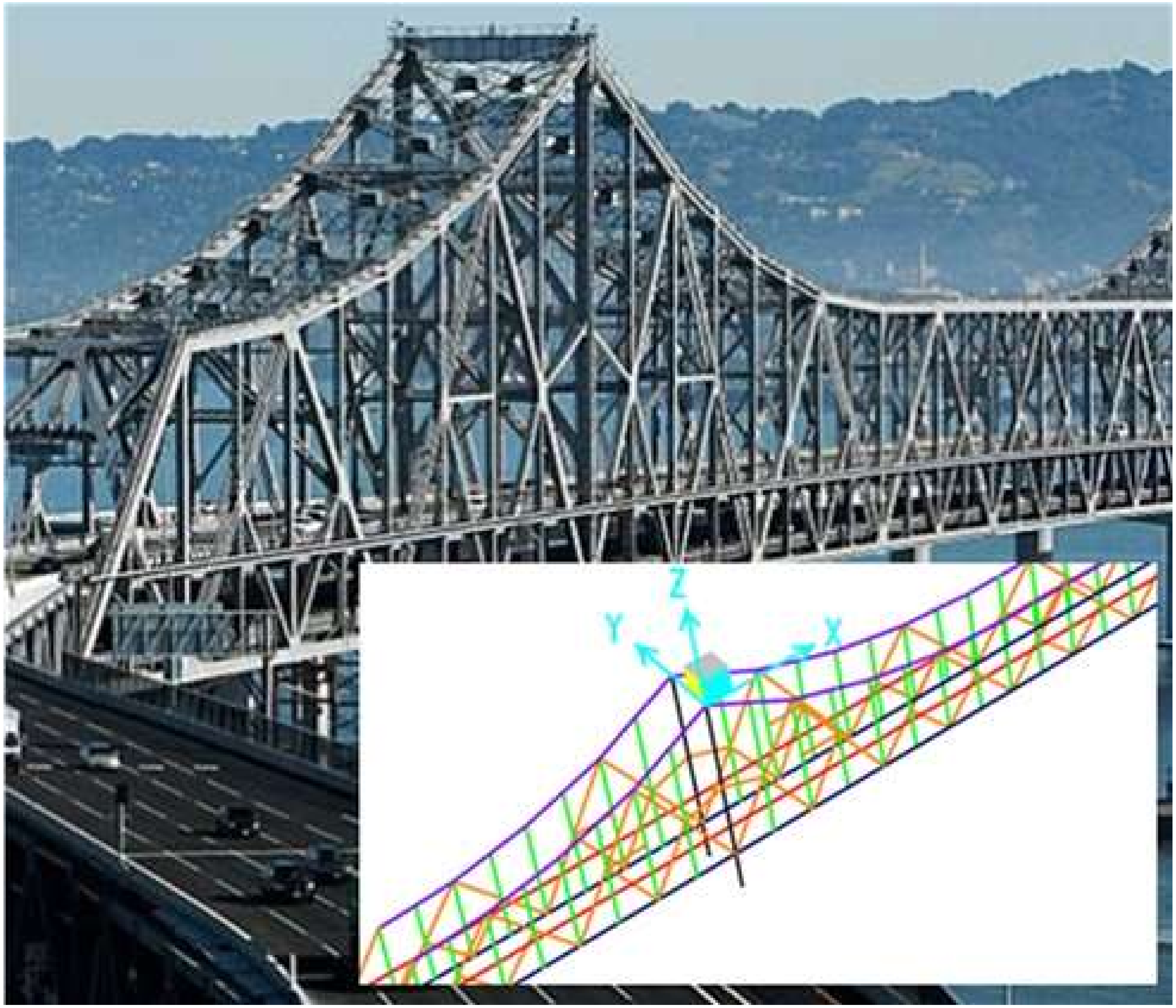


Figure P-257

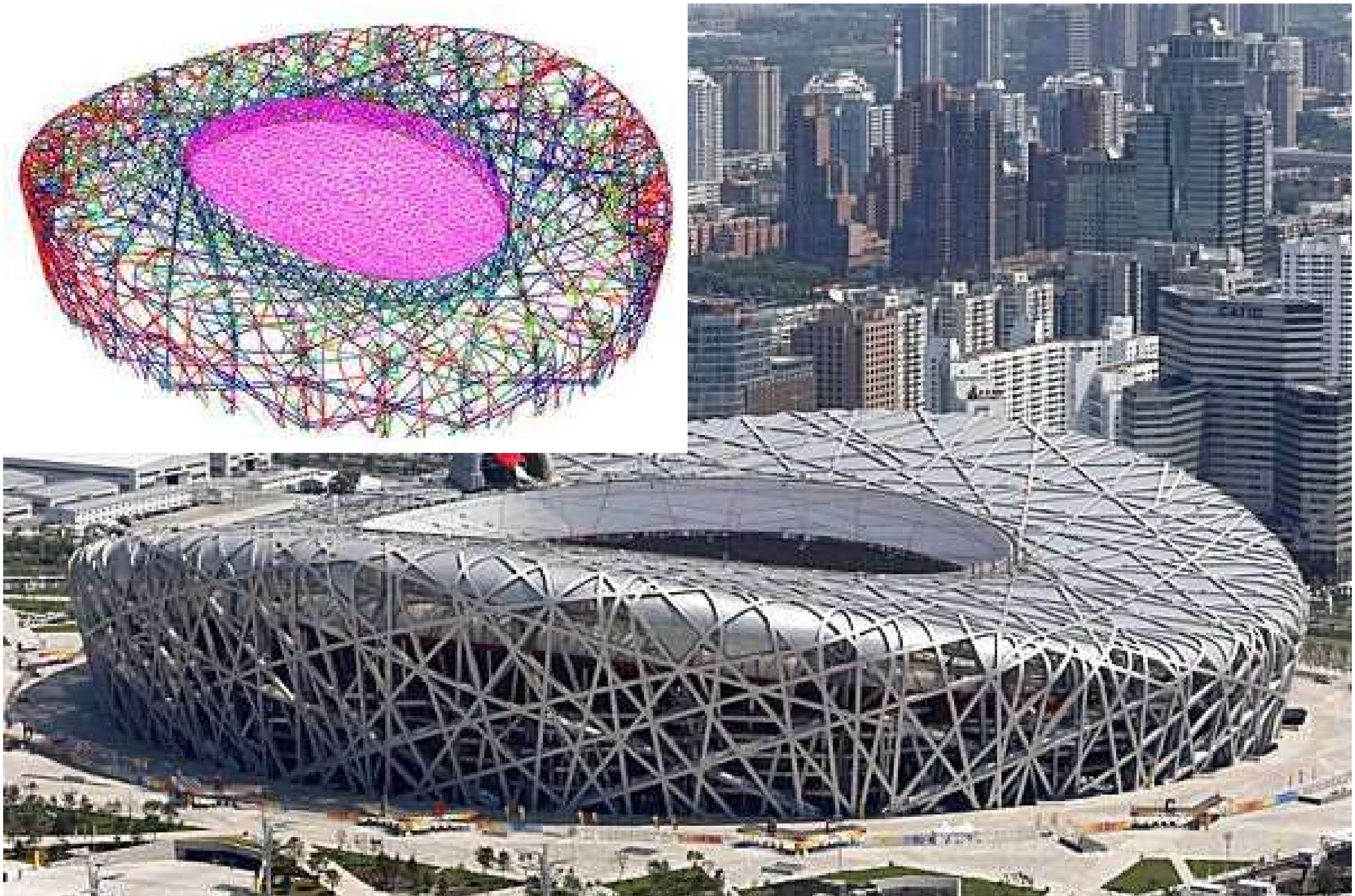


Truss Structure

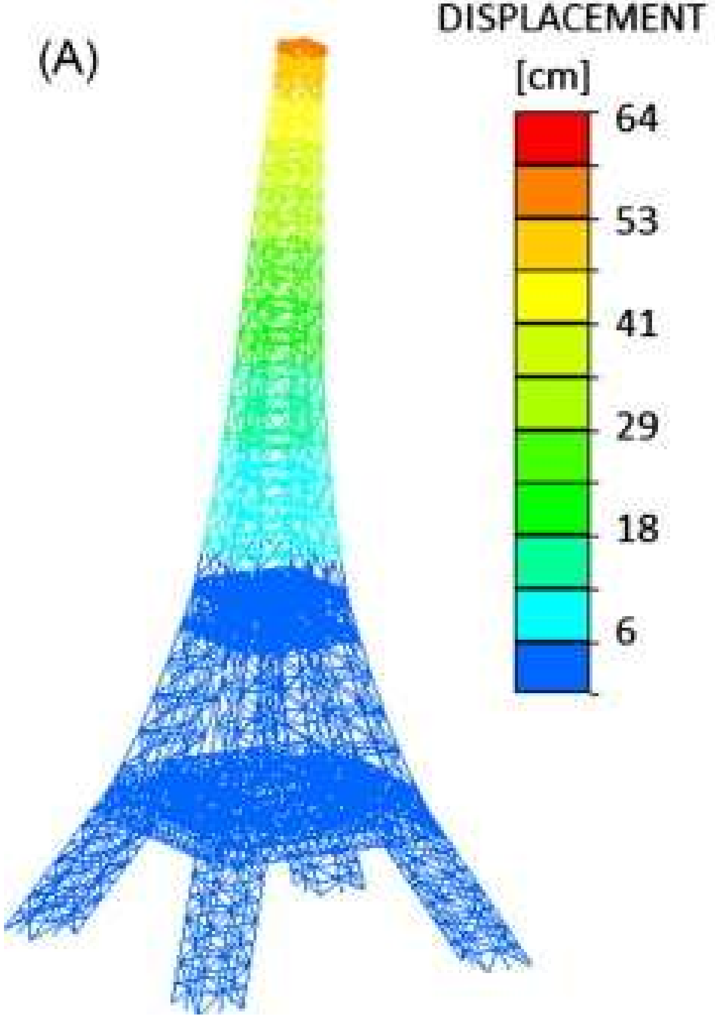


Truss Structure

Truss Structure



Truss Structure



Eigenvalues of Stiffness Matrix

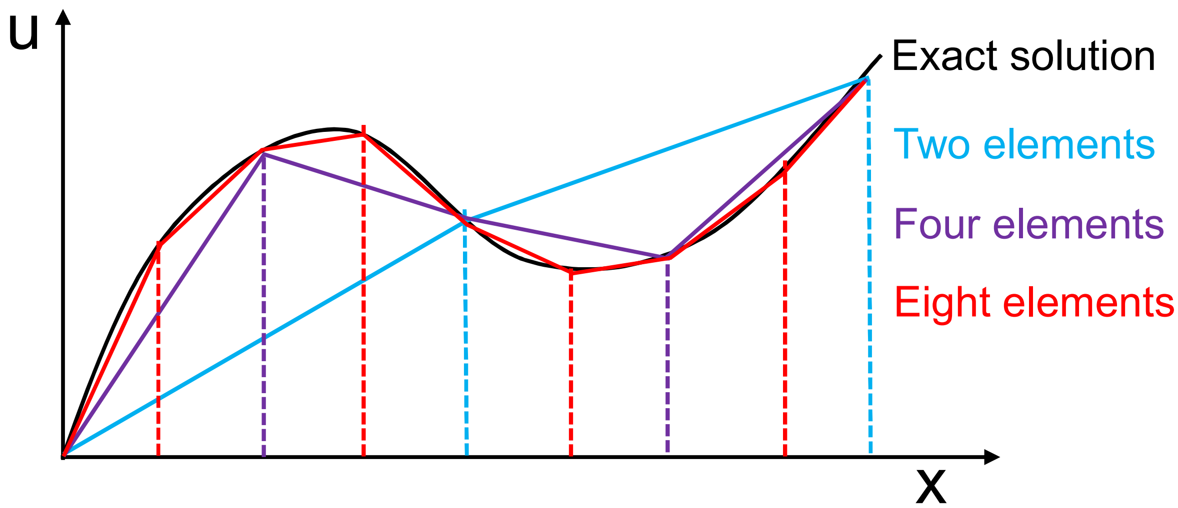
```
% 1D Bar
k1d=[1 -1; -1 1]
lambda = eig(k1d)


% 2D Bar
l=cos(pi/4); m=sin(pi/4);
k2d=[l*l  l*m -l*l -l*m;
     l*m  m*m -l*m -m*m;
    -l*l -l*m  l*l  l*m;
    -l*m -m*m  l*m  m*m]
lambda = eig(k2d)
```

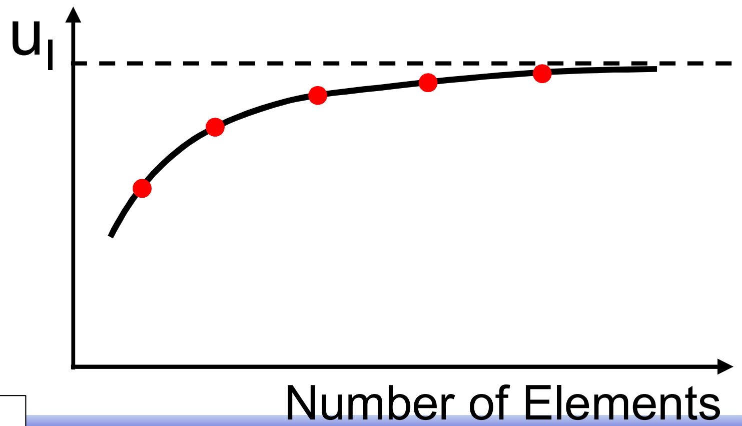
```
k1d =
     1    -1
    -1     1
lambda =
     0
     2

k2d =
    0.5000    0.5000   -0.5000   -0.5000
    0.5000    0.5000   -0.5000   -0.5000
   -0.5000   -0.5000    0.5000    0.5000
   -0.5000   -0.5000    0.5000    0.5000
lambda =
   -0.0000
   -0.0000
    0.0000
    2.0000
```

Thermal Stress and Strain

Interpolation and Convergence

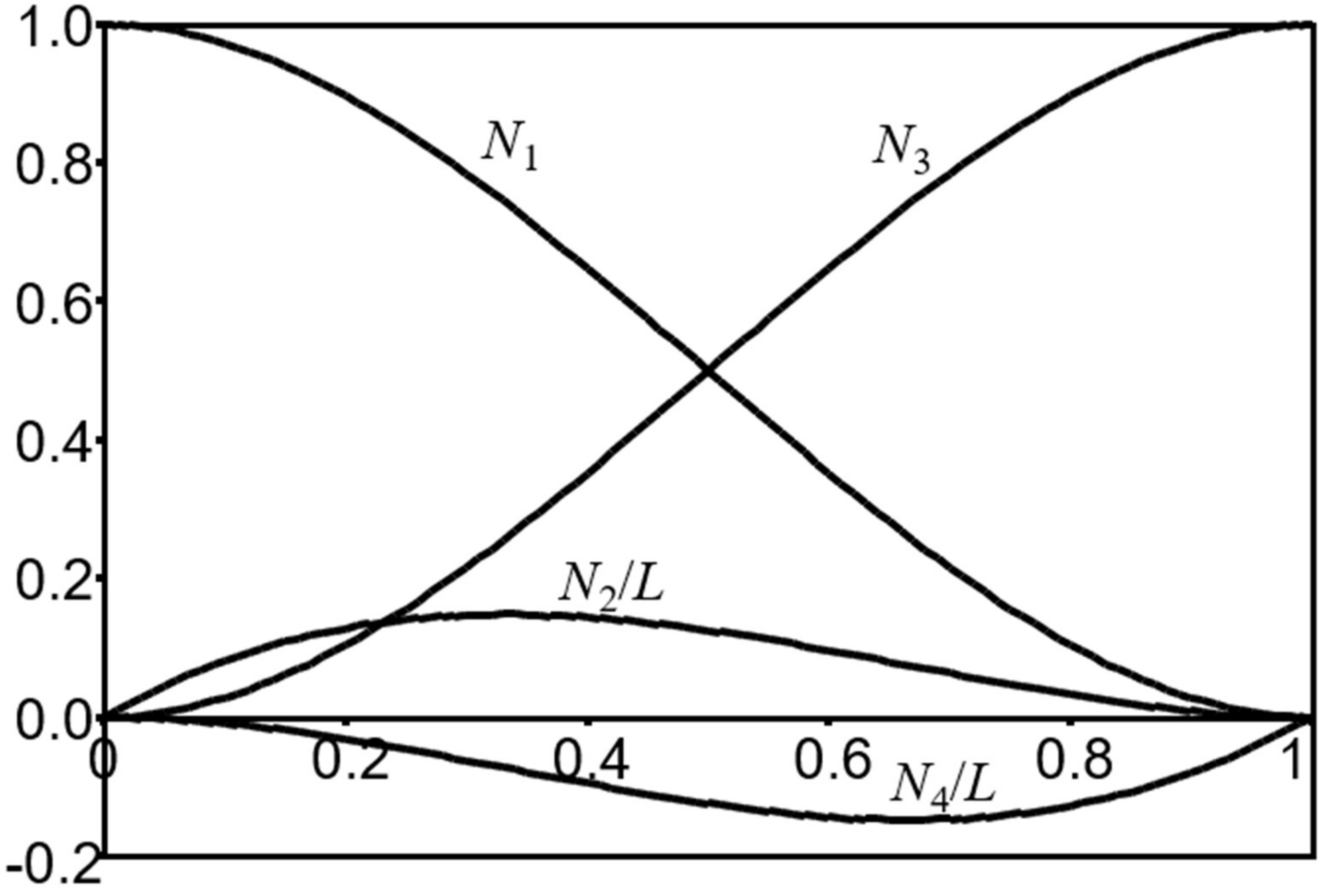
- How do you know the FEM solution is accurate?
- Convergence: the finite element solution converges to the exact solution as the size of elements decreases



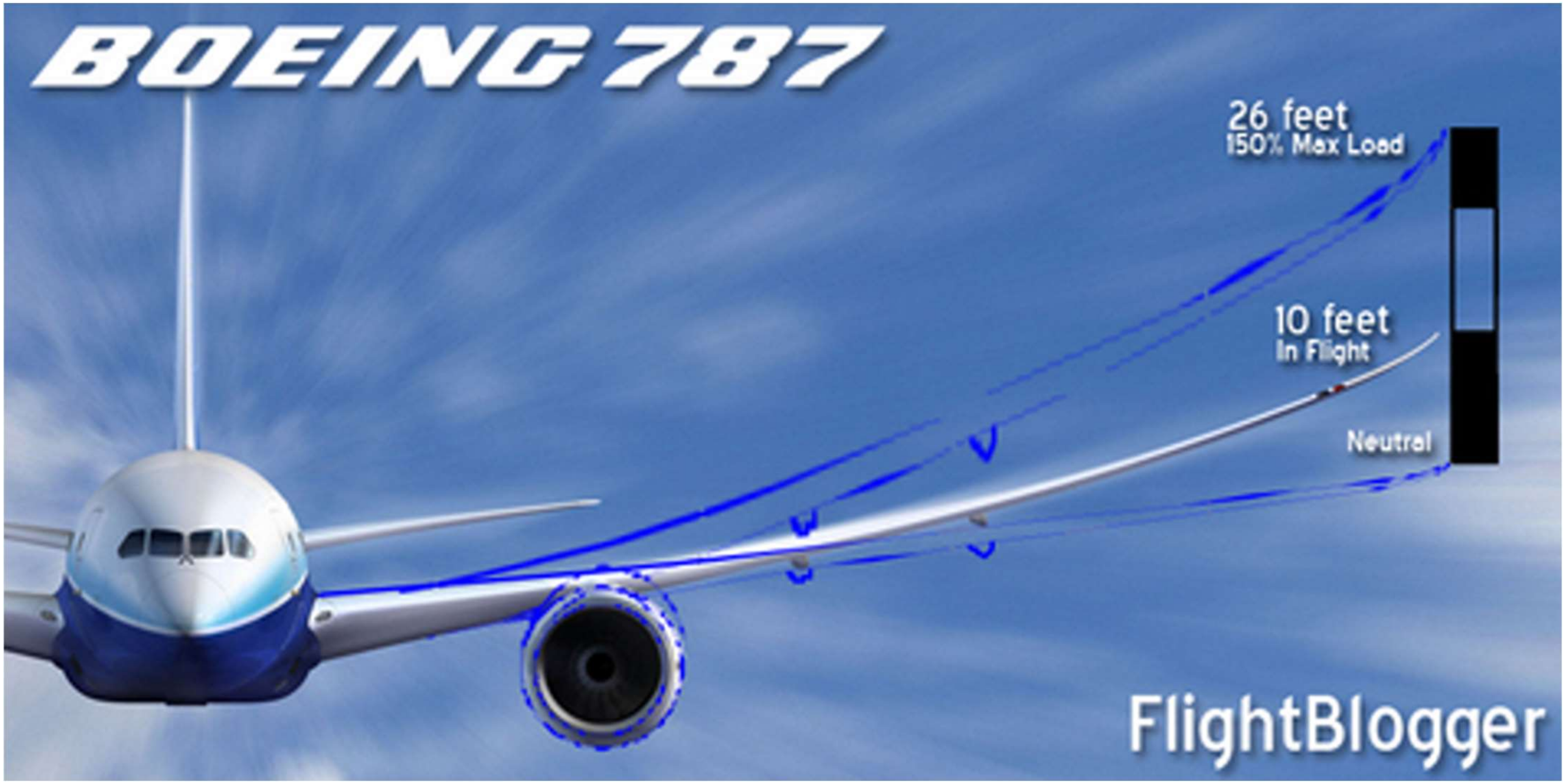


Try with at least three different element sizes to determine convergence

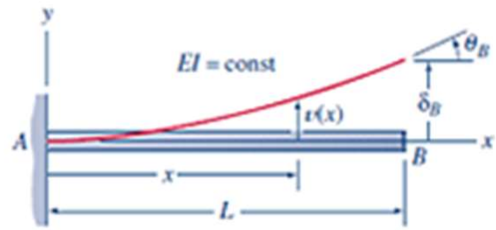
Beam Shape Functions



Wing Deflection of Boeing 787



Beam Deflection Curve 1



Notation

$v(x)$ = deflection in the y direction

$v'(x)$ = slope of the deflection curve

$\delta_B = v(L)$ = deflection at end B

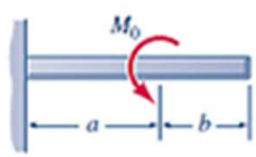
$\theta_B = v'(L)$ = slope at end B

1

$$v = \frac{M_0 x^2}{2EI} \qquad v' = \frac{M_0 x}{EI}$$

$$\delta_B = \frac{M_0 L^2}{2EI} \qquad \theta_B = \frac{M_0 L}{EI}$$

2



$$v = \frac{M_0 x^2}{2EI} \qquad v' = \frac{M_0 x}{EI} \qquad 0 \le x \le a$$

$$v = \frac{M_0 a}{2EI}(2x - a) \qquad v' = \frac{M_0 a}{EI} \qquad a \le x \le L$$

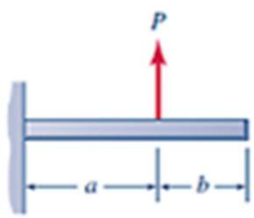
$$\delta_B = \frac{M_0 a}{2EI}(2L - a) \qquad \theta_B = \frac{M_0 a}{EI}$$

3

$$v = \frac{Px^2}{6EI}(3L - x) \qquad v' = \frac{Px}{2EI}(2L - x)$$

$$\delta_B = \frac{PL^3}{3EI} \qquad \theta_B = \frac{PL^2}{2EI}$$

4



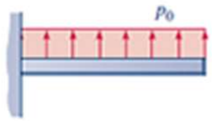
$$v = \frac{Px^2}{6EI}(3a - x) \qquad v' = \frac{Px}{2EI}(2a - x) \qquad 0 \le x \le a$$

$$v = \frac{Pa^2}{6EI}(3x - a) \qquad v' = \frac{Pa^2}{2EI} \qquad a \le x \le L$$

$$\delta_B = \frac{Pa^2}{6EI}(3L - a) \qquad \theta_B = \frac{Pa^2}{2EI}$$

Beam Deflection Curve 2

5

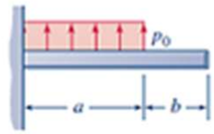


$$v = \frac{p_0 x^2}{24EI}(6L^2 - 4Lx + x^2)$$

$$v' = \frac{p_0 x}{6EI}(3L^2 - 3Lx + x^2)$$

$$\delta_B = \frac{p_0 L^4}{8EI} \qquad \theta_B = \frac{p_0 L^3}{6EI}$$

6



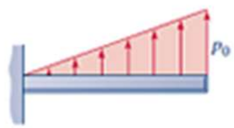
$$v = \frac{p_0 x^2}{24EI}(6a^2 - 4ax + x^2) \qquad 0 \le x \le a$$

$$v' = \frac{p_0 x}{6EI}(3a^2 - 3ax + x^2) \qquad 0 \le x \le a$$

$$v = \frac{p_0 a^3}{24EI}(4x - a) \qquad v' = \frac{p_0 a^3}{6EI} \qquad a \le x \le L$$

$$\delta_B = \frac{p_0 a^3}{24EI}(4L - a) \qquad \theta_B = \frac{p_0 a^3}{6EI}$$

7

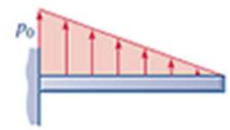


$$v = \frac{p_0 x^3}{120LEI}(20L^3 - 10L^2 x + x^3)$$

$$v' = \frac{p_0 x}{24LEI}(8L^3 - 6L^2 x + x^3)$$

$$\delta_B = \frac{11 p_0 L^4}{120EI} \qquad \theta_B = \frac{p_0 L^3}{8EI}$$

8

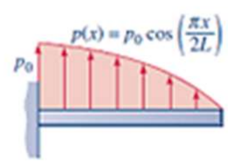


$$v = \frac{p_0 x^2}{120LEI}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$$

$$v' = \frac{p_0 x}{24LEI}(4L^3 - 6L^2 x + 4Lx^2 - x^3)$$

$$\delta_B = \frac{p_0 L^4}{30EI} \qquad \theta_B = \frac{p_0 L^3}{24EI}$$

9



$$v = \frac{p_0 L}{3\pi^4 EI}\left(48L^3 \cos\frac{\pi x}{2L} - 48L^3 + 3\pi^3 Lx^2 - \pi^3 x^3\right)$$

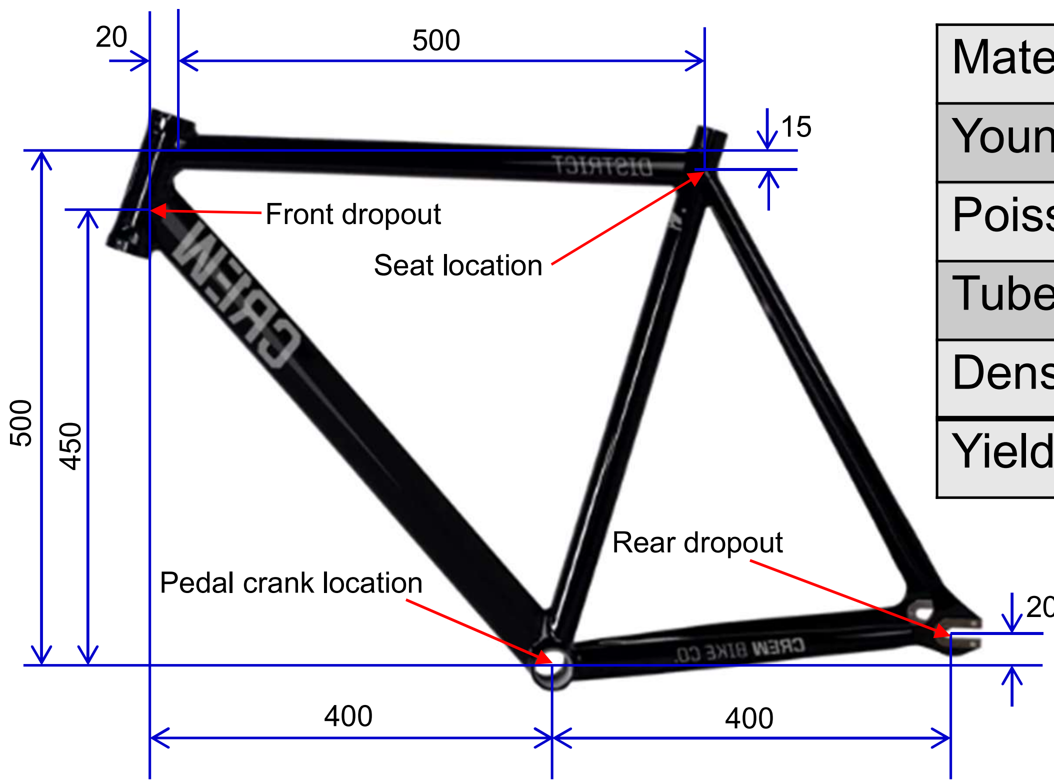
$$v' = \frac{p_0 L}{\pi^3 EI}\left(2\pi^2 Lx - \pi^2 x^2 - 8L^2 \sin\frac{\pi x}{2L}\right)$$

$$\delta_B = \frac{2 p_0 L^4}{3\pi^4 EI}(\pi^3 - 24) \qquad \theta_B = \frac{p_0 L^3}{\pi^3 EI}(\pi^2 - 8)$$

*Beam-deflection theory is covered in Chapter 7. The sign convention used here is the same as in Chapter 7.

Project 1 (Due 10/15) Bicycle Frame Design

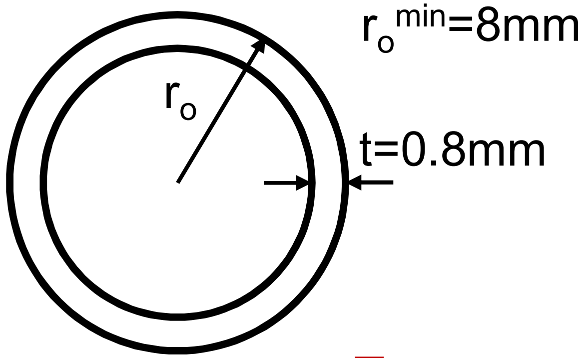
- Aluminum tube diameter design
- Under 2 loading conditions (weight and impact)
- Consider both static and buckling analyses



Material Property	Value
Young's Modulus (E)	70 GPa
Poisson's Ratio (n)	0.33
Tube thickness	0.8 mm
Density (r)	2,580 kg/m^3
Yield Strength (s_Y)	210 MPa

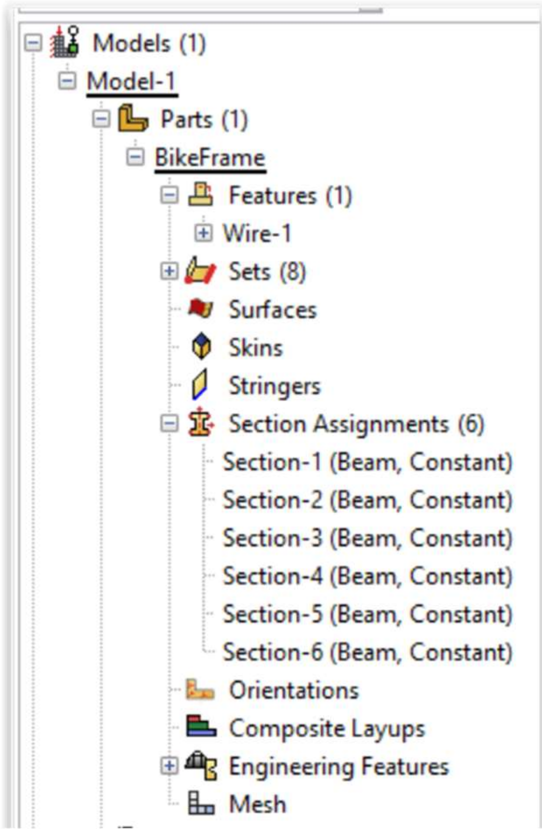
Project 1 Bicycle Frame Design *cont.*

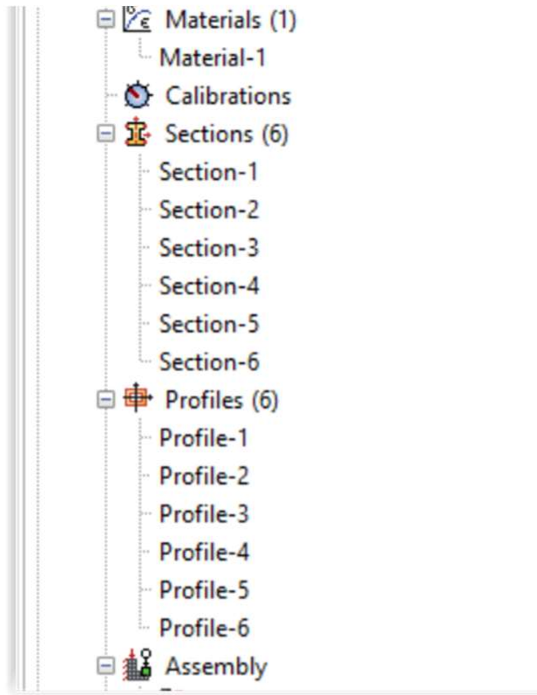
- Goal: Determine the minimum outer diameters of individual tubes such that the bicycle can support loads and safe from buckling
  - Dynamic factor = 3.0
  - Safety factor = 1.5
- Vertical loading condition (downward)
  - 2,000N at the seat
  - 600N at the pedal
- Impact loading condition (horizontal)
  - 1,500N at the front dropout

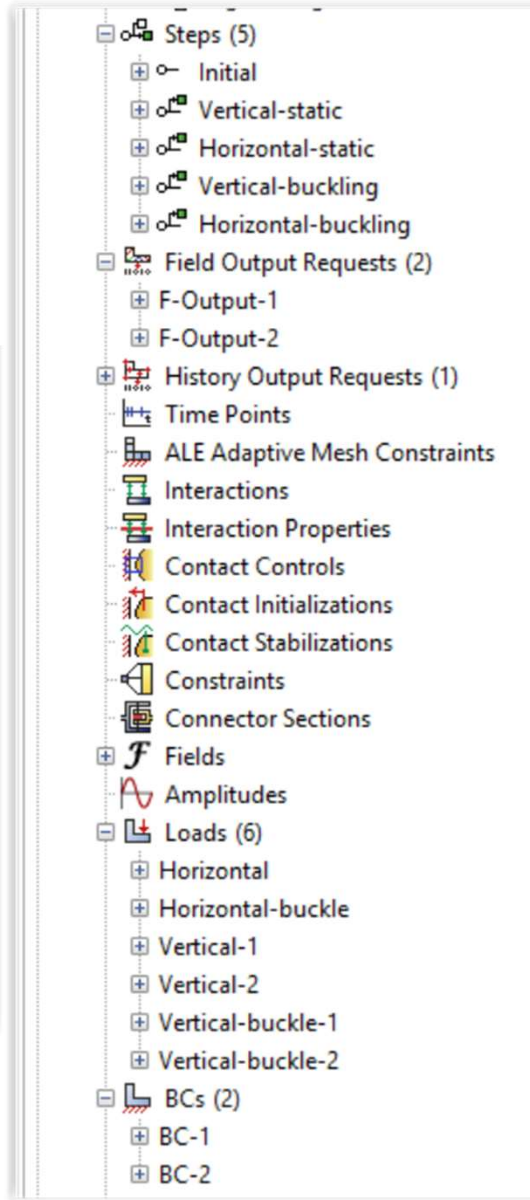


Abaqus Modeling

- 6 tubes: 6 sections, 6 profiles
- 4 Steps (2 static steps, 2 buckling steps)
- 5 beam elements per tube (2 elements for dropout)





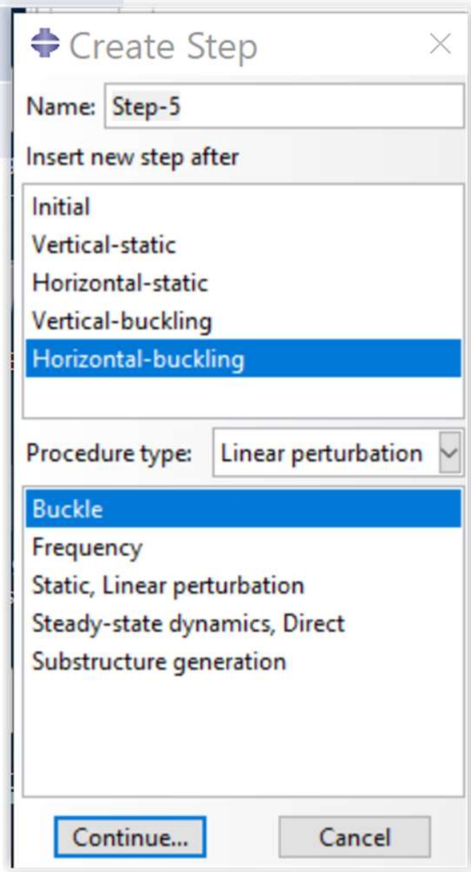


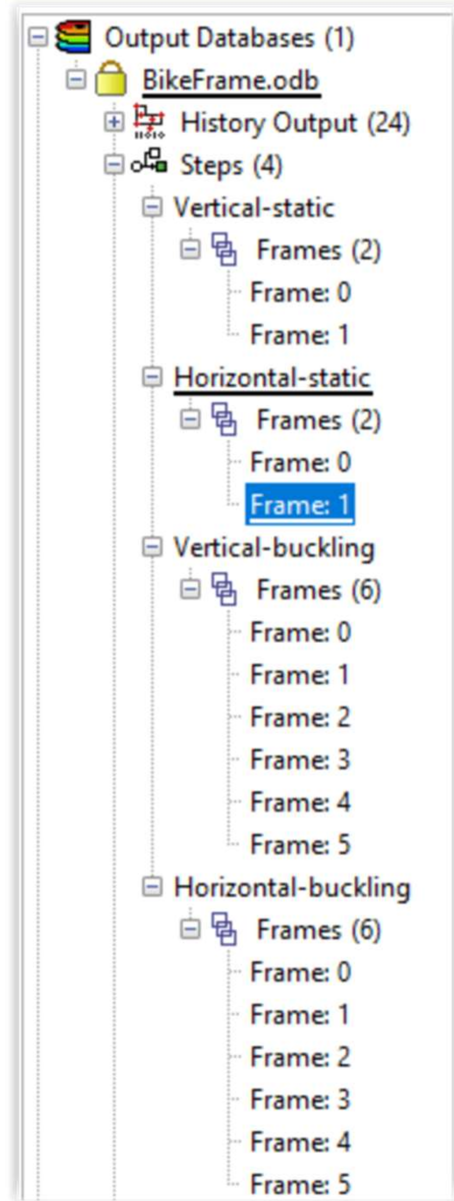
Abaqus Modeling

- Use mm, N unit

Unit	MKS	N, mm, sec
Length	1 m	1000 mm
Force	1 N	1 N
Stress	1 MPa	
Density	1 kg/m^3	

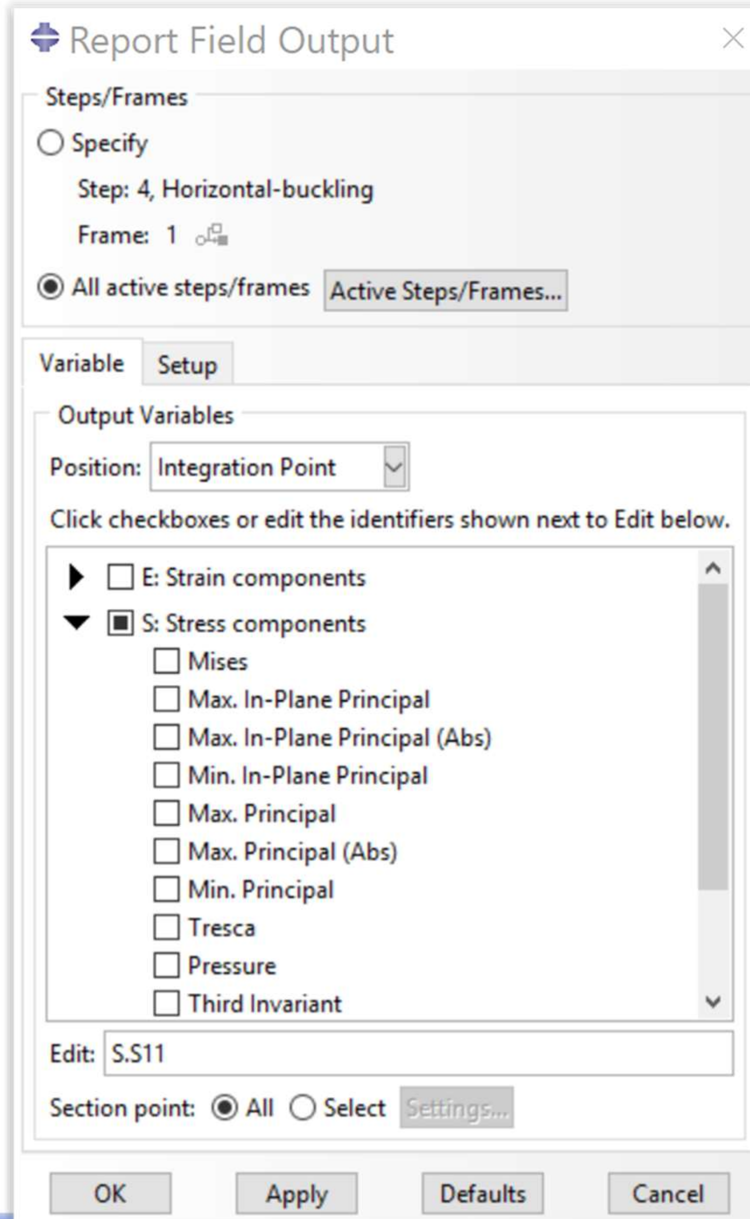
- Linear perturbation
  – Static, Linear perturbation
  – Buckling





Abaqus Modeling

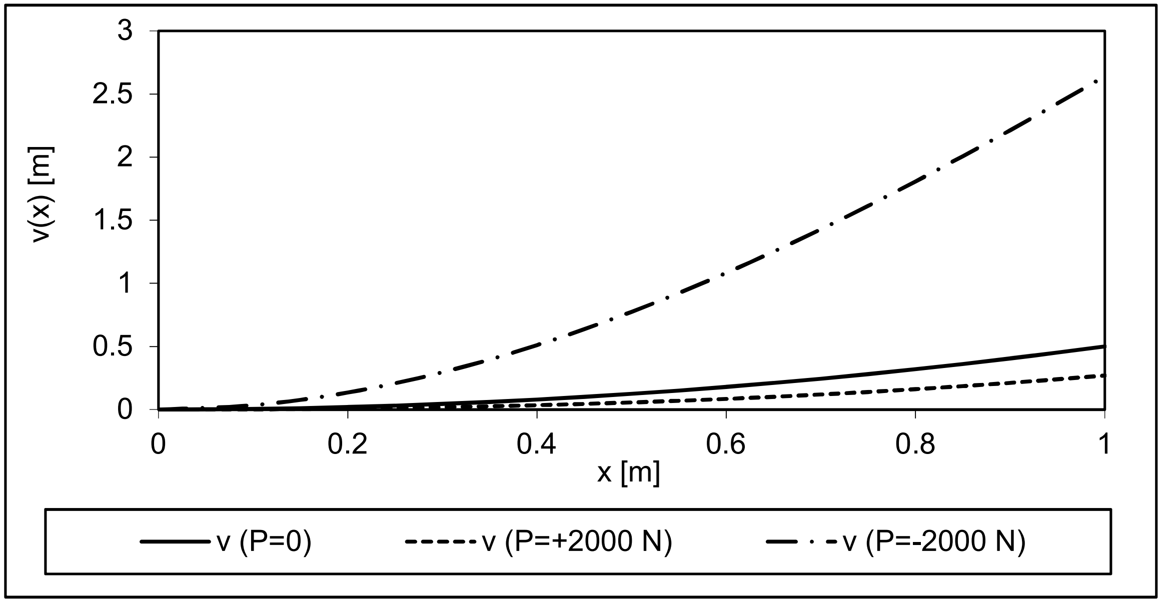
- Show Abaqus.rpt file
- Understand integration points
- Use either min or max stress



Project 1 Bicycle Frame Design *cont.*

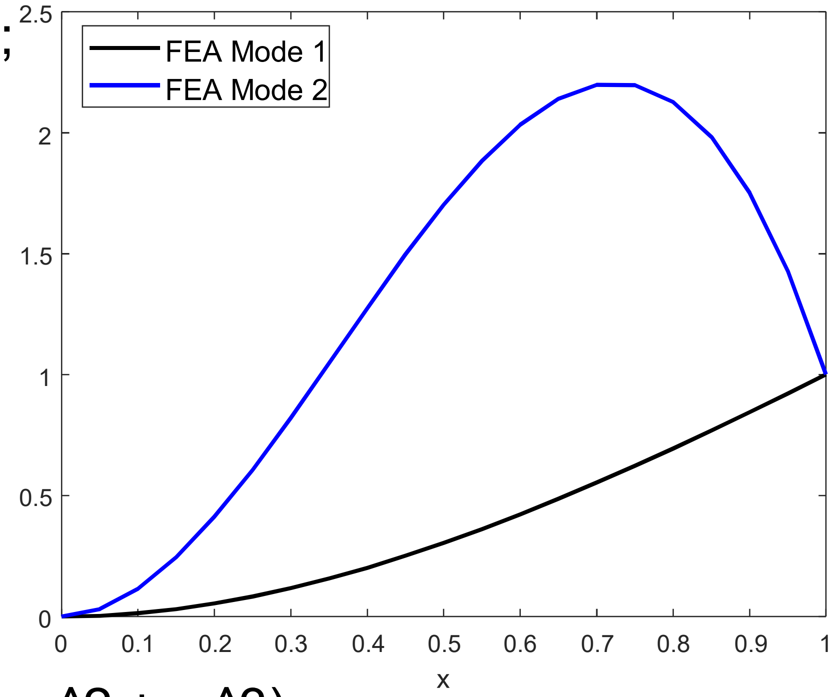
- Report in the conference paper format (template on Canvas)
  - Report should be readable and complete by itself
  - Include introduction, approach, assumptions, results, conclusion, discussion, and references
- Report must include the following information for each load case:
  - plot of FE geometry with node/element labels and boundary/load conditions
  - a table of design iterations that shows the maximum stresses, tube diameters, and the weight of the bike
  - bending moment diagram and von Mises stress plot at the final design for two static cases
  - plot of deformed geometry with the magnification factor for two static cases
  - the first mode shape and critical load for two buckling load cases
- Report must be less than 10 pages (PDF file).
  - also submit your CAE and ODB files
  - All results must be analyzed and summarized

Beam Deflection with Axial Force



Column Buckling, One-element, Clamped

- A=[12 -6;-6 4]; B=[2.4 -0.2;-0.2 0.266];
- [v e]=eig(A,B)
- v = [-0.6420 -0.1800
  -1.0062 1.7315]
- e = [1.2438 0
  0 16.1225]
- s=0:0.05:1;
- v1=-0.6420*(3*s.^2 - 2*s.^3)-1.0062*(-s.^2 + s.^3);
- v2=-0.1800*(3*s.^2 - 2*s.^3)+1.7315*(-s.^2 + s.^3);
- v1=-v1/0.6420; v2=-v2/0.18; %Normalization
- plot(s,v1,'-k',s,v2,'-b');

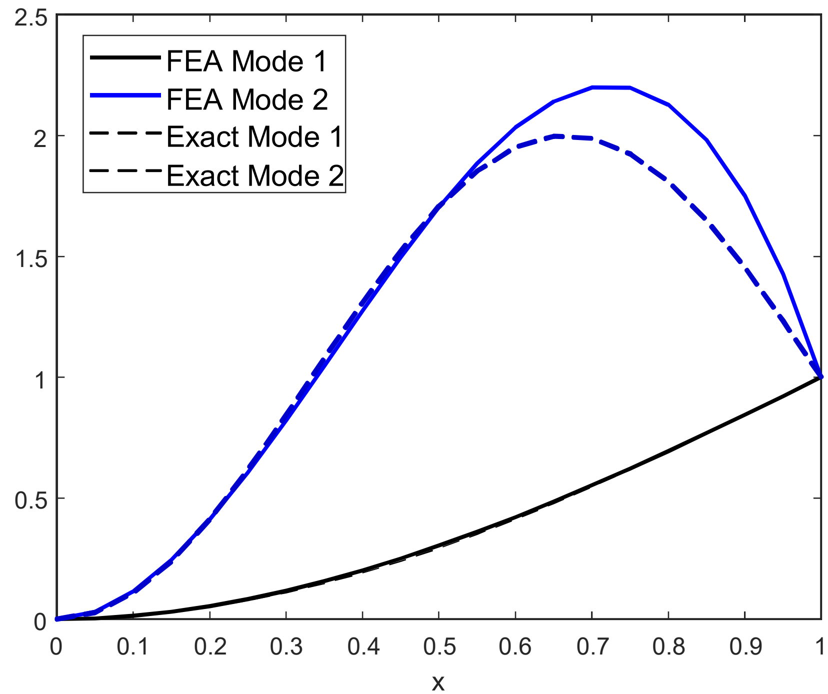


Column Buckling

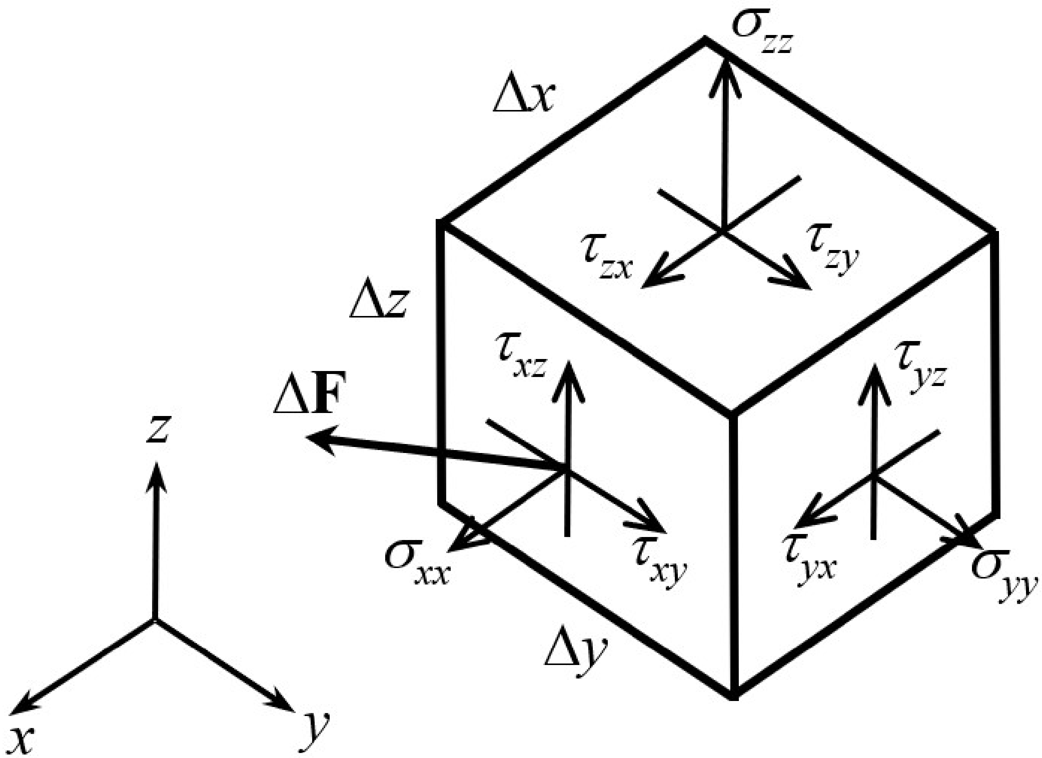
- s=0:0.05:1;
- va1=1-cos(pi/2*s); va2=1-cos(3*pi/2*s);
- plot(s,v1,'-k',s,v2,'-b',s,va1,'--k',s,va2,'--b');
- legend('FEA Mode 1','FEA Mode 2','Exact Mode 1','Exact Mode 2')

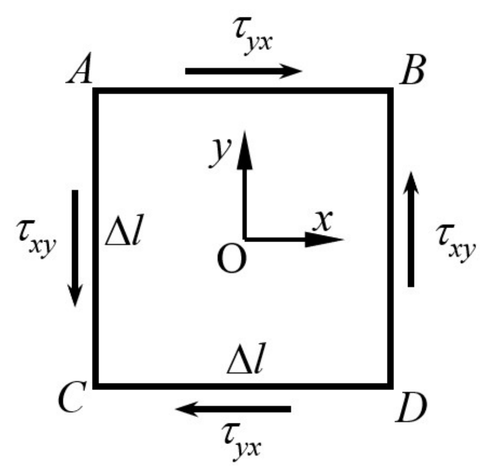
- $P_{cr1exact}$ = 2467.4 N
- $P_{cr2exact}$ = 22,207 N
- P_{cr1FEA} = 2487 N
- P_{cr2FEA} = 32,245 N

- How to improve FEA results?

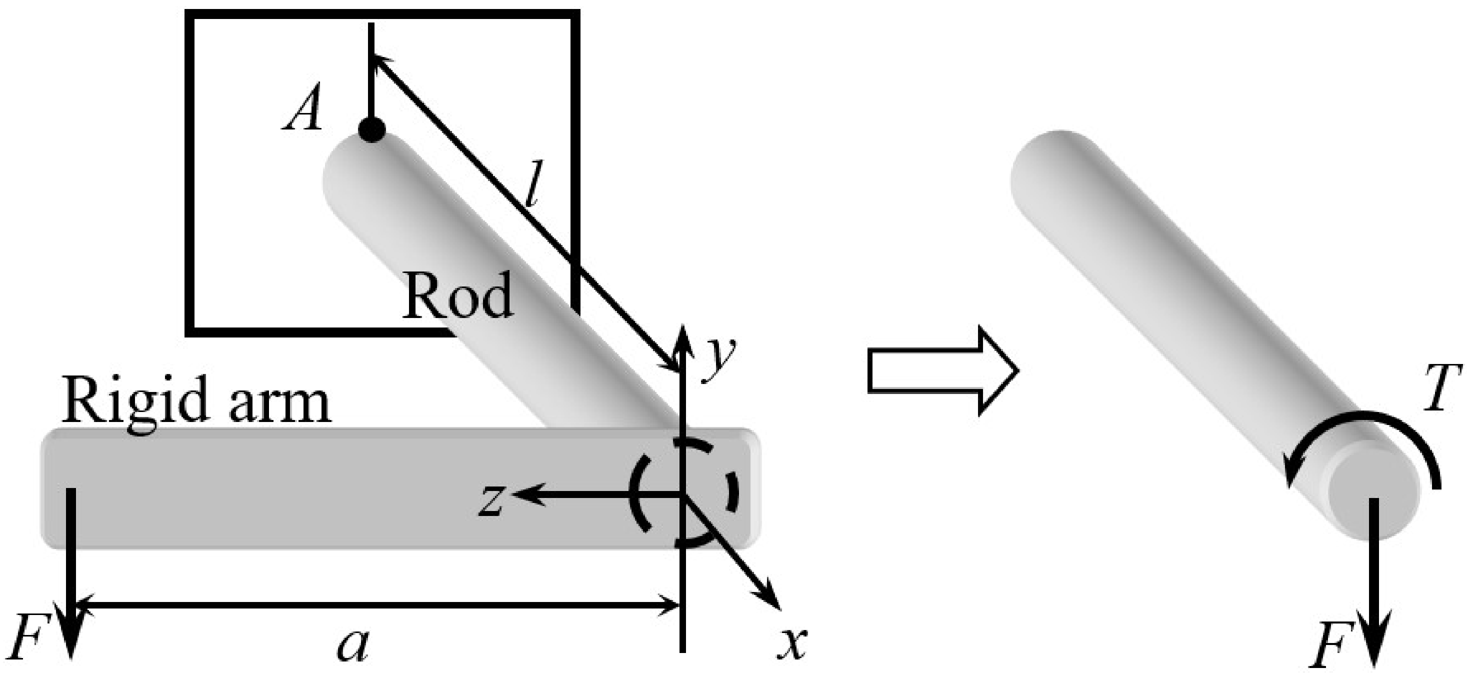


Cartesian Components of Stress

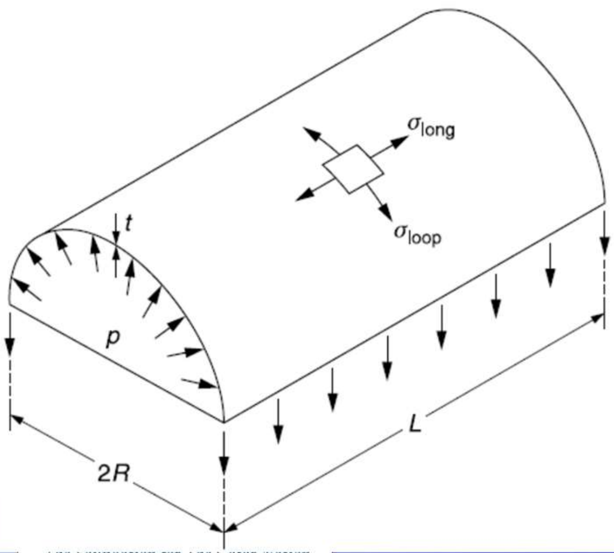




Example



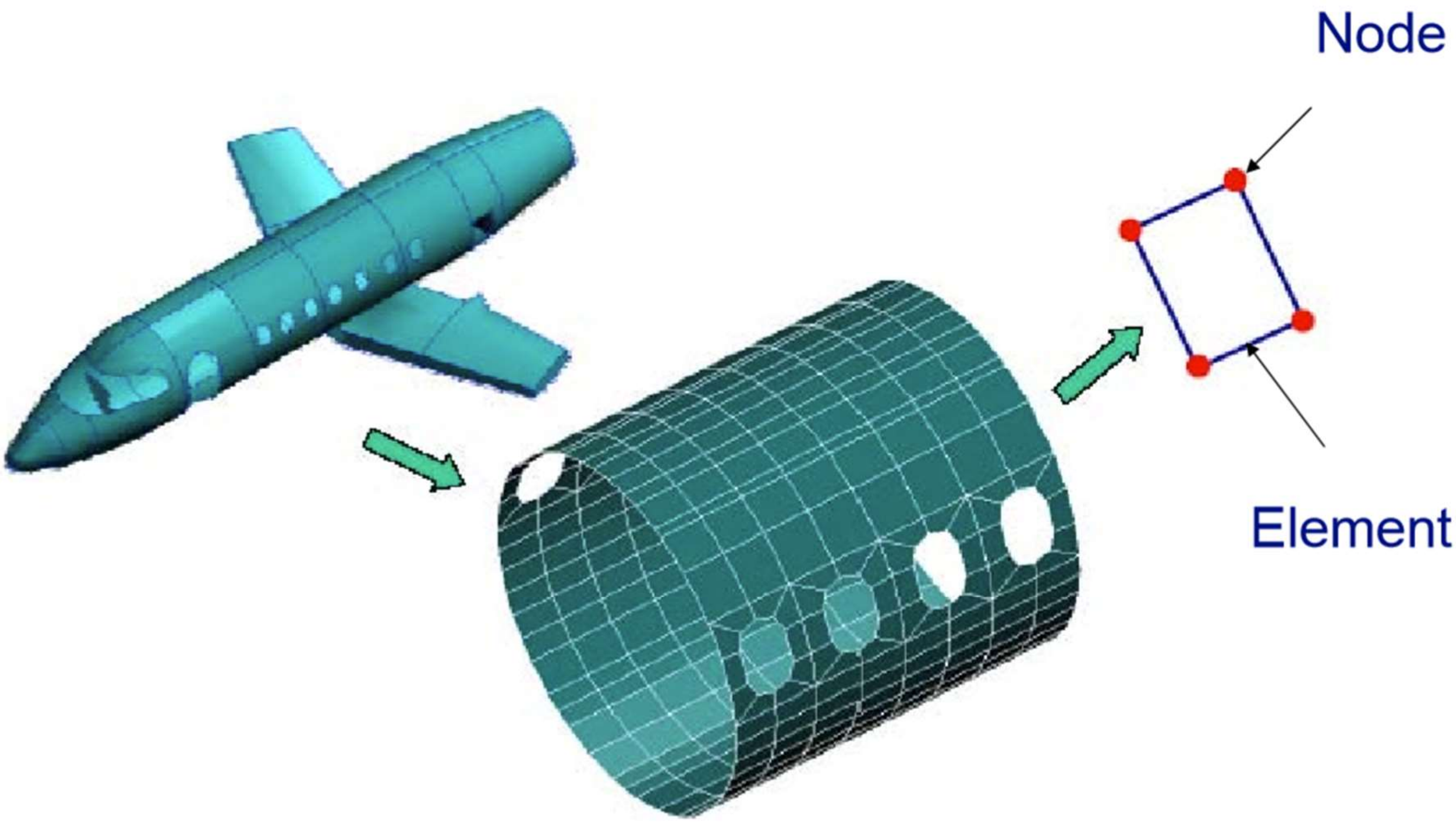
Plane solids: CST

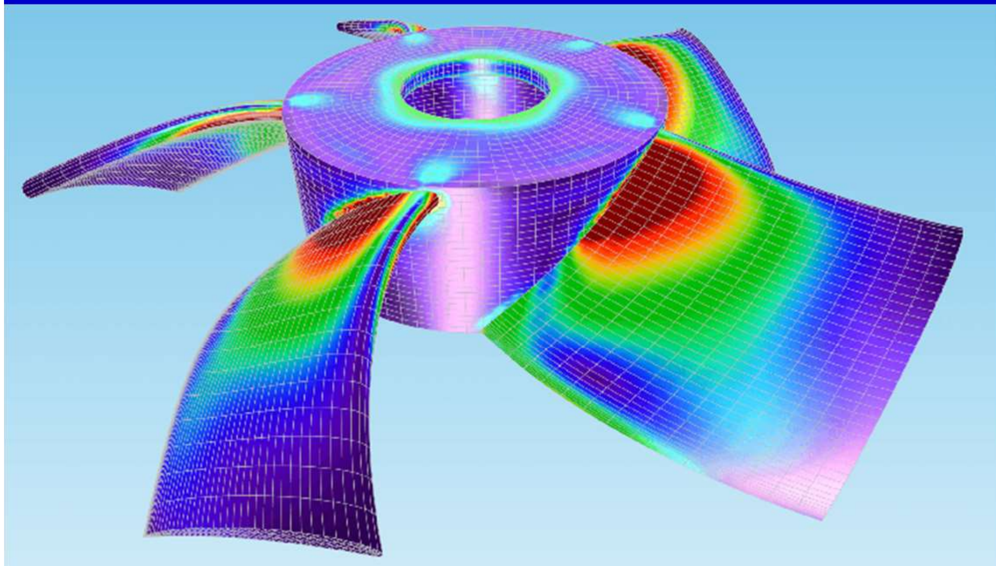


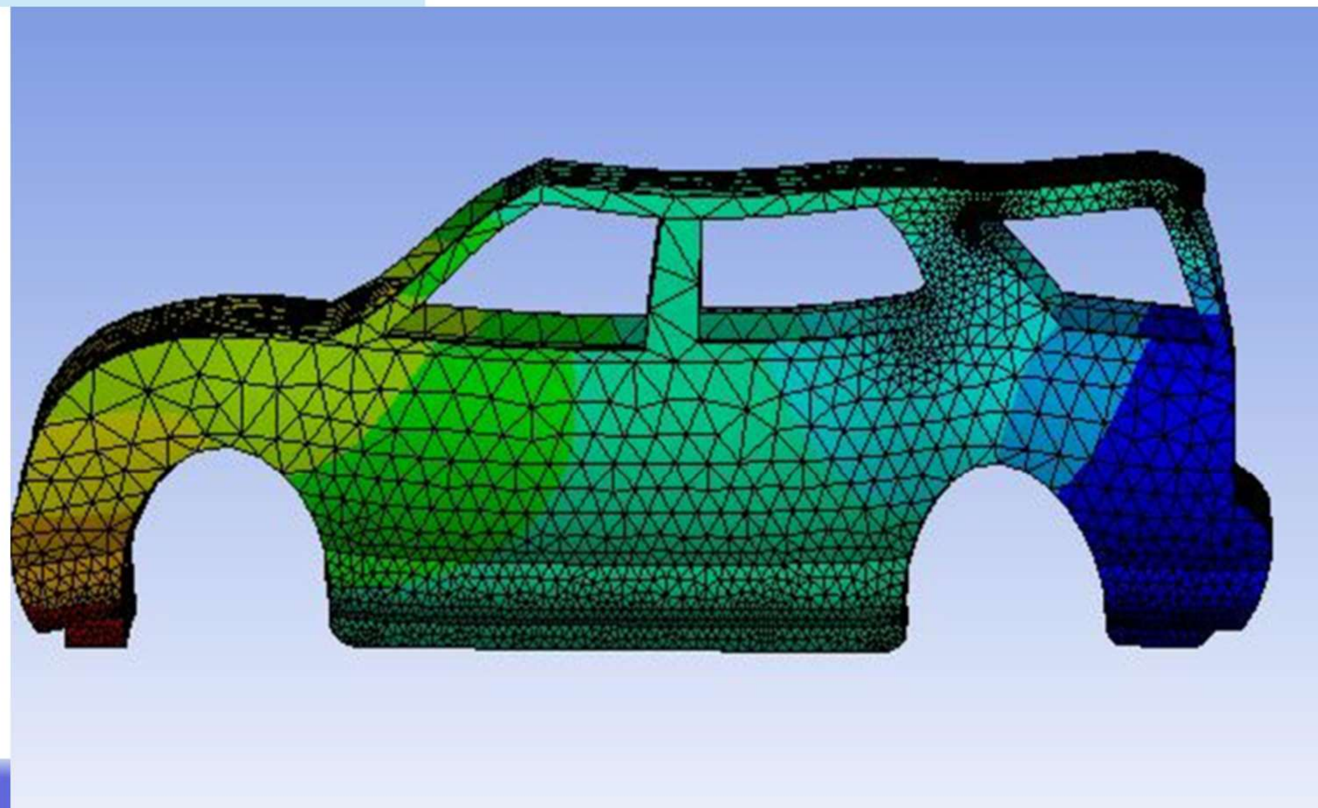
For B-737: r= 6 ft= 72” t= 0.063”

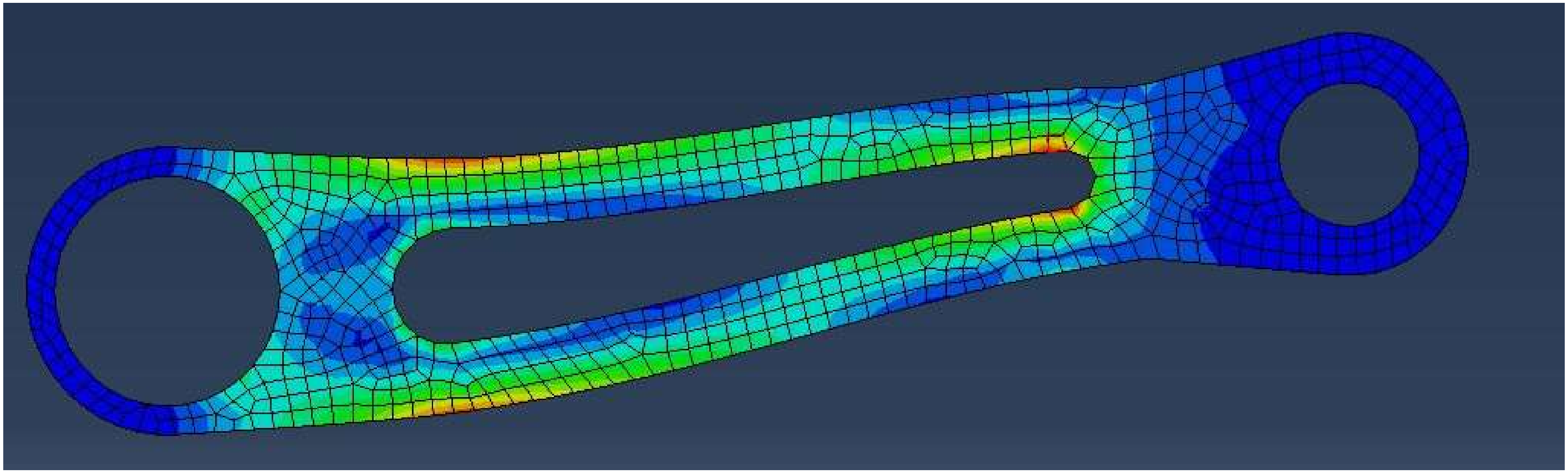
$$\sigma_{hoop} = \frac{pr}{t}$$ =1143p

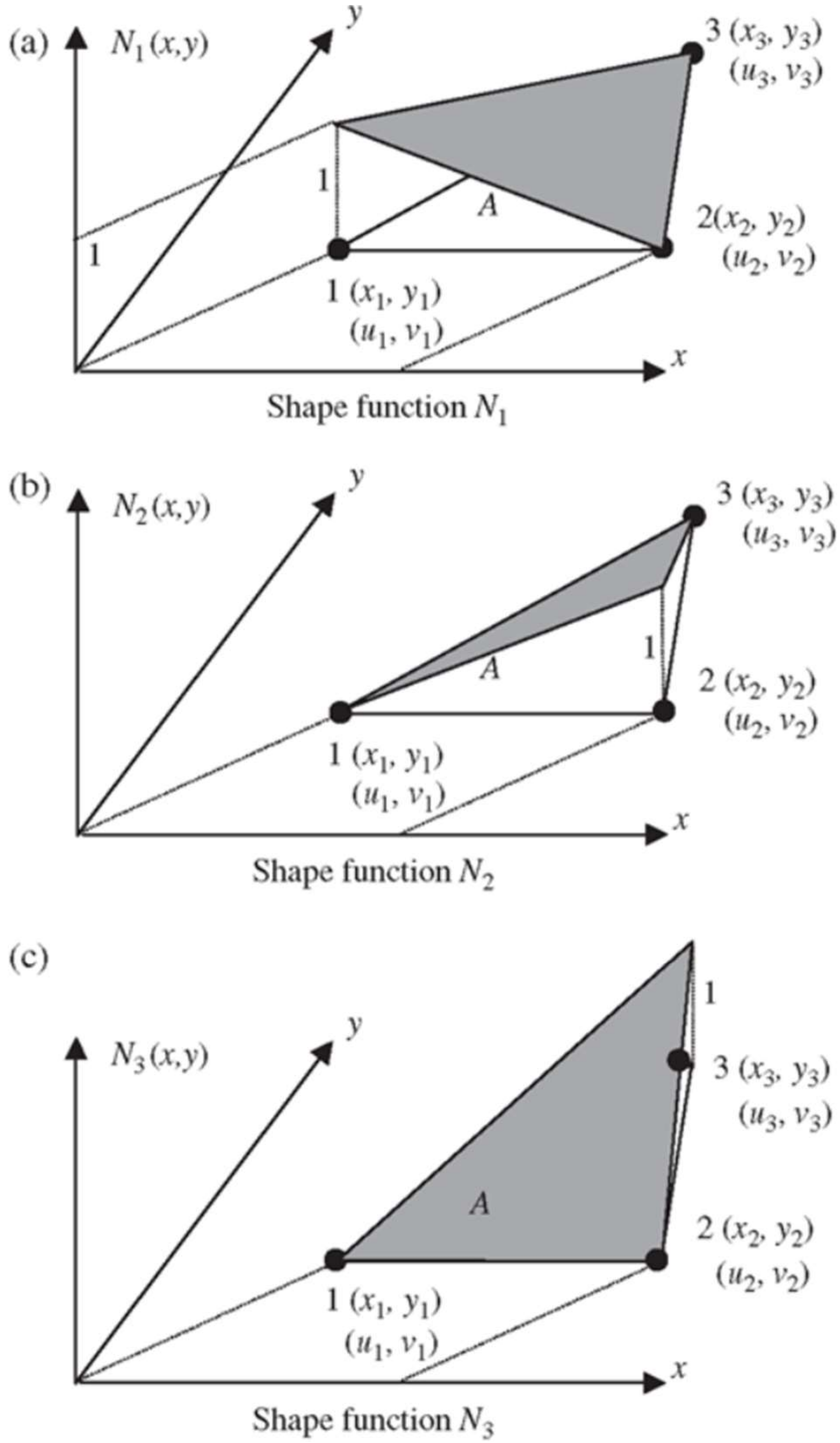
$$\sigma_{long} = \frac{pr}{2t}$$ =571p


Node
Element









$$N_1(x_1, y_1) = 1$$
$$N_1(x_2, y_2) = 0$$
$$N_1(x_3, y_3) = 0$$

$$N_2(x_1, y_1) = 0$$
$$N_2(x_2, y_2) = 1$$
$$N_2(x_3, y_3) = 0$$

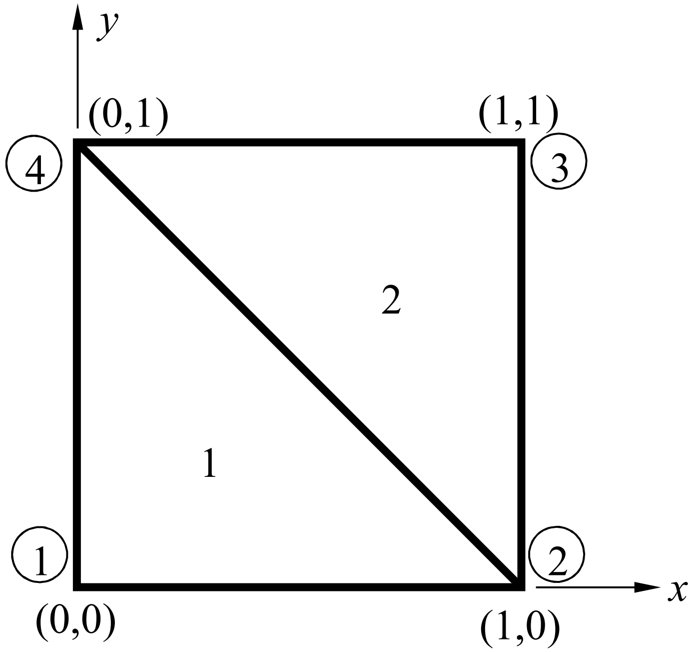
$$N_3(x_1, y_1) = 0$$
$$N_3(x_2, y_2) = 0$$
$$N_3(x_3, y_3) = 1$$

Example 6.1 (Bending)

nodal displacements

$\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{-0.1, 0, 0.1, 0, -0.1, 0, 0.1, 0\}$

Calculate displacements and strains in both elements.



Element 1: Nodes 1-2-4

$$x_1 = 0 \quad y_1 = 0$$
$$x_2 = 1 \quad y_2 = 0$$
$$x_3 = 0 \quad y_3 = 1$$

$$f_1 = 1 \quad f_2 = 0 \quad f_3 = 0$$
$$b_1 = -1 \quad b_2 = 1 \quad b_3 = 0$$
$$c_1 = -1 \quad c_2 = 0 \quad c_3 = 1$$

$$N_1(x,y) = 1 - x - y$$
$$N_2(x,y) = x$$
$$N_3(x,y) = y$$

$$u^{(1)}(x,y) = \sum_{I=1}^{3} N_I(x,y)u_I = 0.1(2x + 2y - 1)$$

$$v^{(1)}(x,y) = \sum_{I=1}^{3} N_I(x,y)v_I = 0.0$$

$$\varepsilon_{xx}^{(1)} = \frac{\partial u^{(1)}}{\partial x} = 0.2$$

$$\varepsilon_{yy}^{(1)} = \frac{\partial v^{(1)}}{\partial y} = 0.0$$

$$\gamma_{xy}^{(1)} = \frac{\partial u^{(1)}}{\partial y} + \frac{\partial v^{(1)}}{\partial x} = 0.2$$

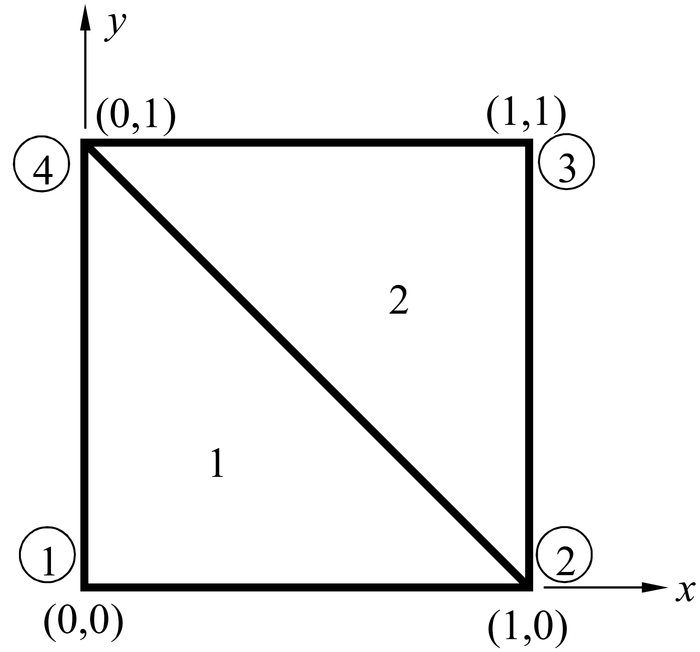
Example 6.1 (Bending) cont.

Element 2: Nodes 2-3-4

$$x_1 = 1 \qquad y_1 = 0$$
$$x_2 = 1 \qquad y_2 = 1$$
$$x_3 = 0 \qquad y_3 = 1$$

$$f_1 = 1 \qquad f_2 = -1 \qquad f_3 = 1$$
$$b_1 = 0 \qquad b_2 = 1 \qquad b_3 = -1$$
$$c_1 = -1 \qquad c_2 = 1 \qquad c_3 = 0$$

$$N_1(x,y) = 1 - y$$
$$N_2(x,y) = x + y - 1$$
$$N_3(x,y) = 1 - x$$



$$\varepsilon_{xx}^{(2)} = \frac{\partial u^{(2)}}{\partial x} = -0.2$$

$$u^{(2)}(x,y) = \sum_{i=1}^{3} N_i(x,y)u_i = 0.1(3 - 2x - 2y)$$

$$\varepsilon_{yy}^{(2)} = \frac{\partial v^{(2)}}{\partial y} = 0.0$$

$$v^{(2)}(x,y) = \sum_{i=1}^{3} N_i(x,y)v_i = 0.0$$

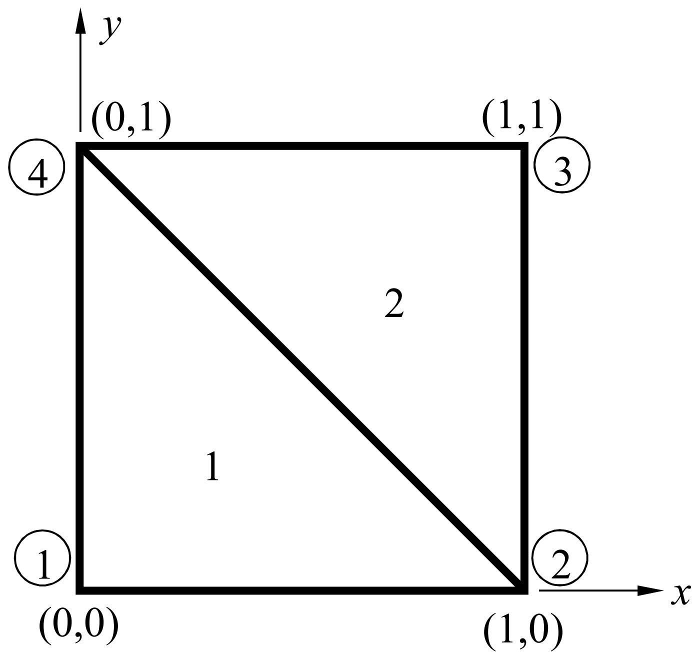
$$\gamma_{xy}^{(2)} = \frac{\partial u^{(2)}}{\partial y} + \frac{\partial v^{(2)}}{\partial x} = -0.2$$

Example 6.1 (Bending) cont.

nodal displacements

$\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{-0.1, 0, 0.1, 0, -0.1, 0, 0.1, 0\}$

Calculate displacements and strains in both elements.



$$\varepsilon_{xx}^{(1)} = \frac{\partial u^{(1)}}{\partial x} = 0.2$$

$$\varepsilon_{yy}^{(1)} = \frac{\partial v^{(1)}}{\partial y} = 0.0$$

$$\gamma_{xy}^{(1)} = \frac{\partial u^{(1)}}{\partial y} + \frac{\partial v^{(1)}}{\partial x} = 0.2$$

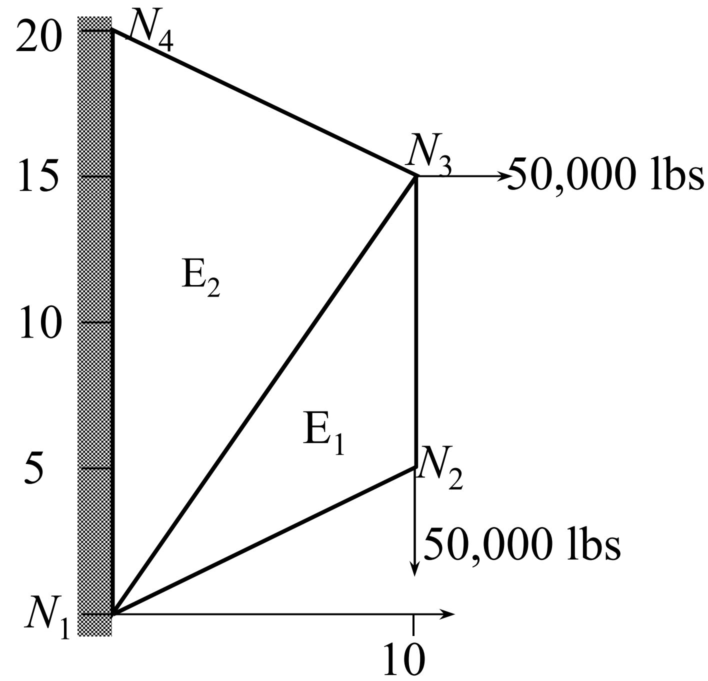
$$\varepsilon_{xx}^{(2)} = \frac{\partial u^{(2)}}{\partial x} = -0.2$$

$$\varepsilon_{yy}^{(2)} = \frac{\partial v^{(2)}}{\partial y} = 0.0$$

$$\gamma_{xy}^{(2)} = \frac{\partial u^{(2)}}{\partial y} + \frac{\partial v^{(2)}}{\partial x} = -0.2$$

Example 6.2

- Cantilevered Plate
  - Thickness h = 0.1 in, $E = 30\times10^6$ psi and $\nu = 0.3$.



- Element 1
  - Area = 0.5×10×10 = 50.

$$x_1 = 0,\ y_1 = 0 \qquad x_2 = 10,\ y_2 = 5 \qquad x_3 = 10,\ y_3 = 15$$

$$b_1 = y_2 - y_3 = -10 \qquad b_2 = y_3 - y_1 = 15 \qquad b_3 = y_1 - y_2 = -5$$

$$c_1 = x_3 - x_2 = 0 \qquad c_2 = x_1 - x_3 = -10 \qquad c_3 = x_2 - x_1 = 10$$

Example 6.2 cont.

$$[\mathbf{B}] = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} -10 & 0 & 15 & 0 & -5 & 0 \\ 0 & 0 & 0 & -10 & 0 & 10 \\ 0 & -10 & -10 & 15 & 10 & -5 \end{bmatrix}$$

$$[\mathbf{C}_\sigma] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} = 3.297\times 10^7\begin{bmatrix} 1 & .3 & 0 \\ .3 & 1 & 0 \\ 0 & 0 & .35 \end{bmatrix}$$

$$[\mathbf{k}^{(1)}] = hA[\mathbf{B}]^T[\mathbf{C}_\sigma][\mathbf{B}] = 3.297\times 10^6\begin{bmatrix} .5 & 0. & -.75 & .15 & .25 & -.15 \\ & .175 & .175 & -.263 & -.175 & .088 \\ & & 1.3 & -.488 & -.55 & .313 \\ & & & .894 & .338 & -.631 \\ & & & & .3 & -.163 \\ & & & & & .544 \end{bmatrix}$$

Example 6.2 cont.

Element 2: Nodes 1-3-4

$$x_1 = 0, y_1 = 0 \qquad x_2 = 10, y_2 = 15 \qquad x_3 = 0, y_3 = 20$$

$$b_1 = y_2 - y_3 = -5 \qquad b_2 = y_3 - y_1 = 20 \qquad b_3 = y_1 - y_2 = -15$$

$$c_1 = x_3 - x_2 = -10 \qquad c_2 = x_1 - x_3 = 0 \qquad c_3 = x_2 - x_1 = 10$$

$$[\mathbf{B}] = \frac{1}{200}\begin{bmatrix} -5 & 0 & 20 & 0 & -15 & 0 \\ 0 & -10 & 0 & 0 & 0 & 10 \\ -10 & -5 & 0 & 20 & 10 & -15 \end{bmatrix}$$

$$[\mathbf{k}^{(2)}] = 3.297 \times 10^6 \begin{bmatrix} .15 & .081 & -.25 & -.175 & .1 & .094 \\ & .272 & -.15 & -.088 & .069 & -.184 \\ & & 1. & 0. & -.75 & .15 \\ & & & .35 & .175 & -.263 \\ & & & & .65 & -.244 \\ & & & & & .447 \end{bmatrix}$$

Example 6.2 cont.

- Assembly

$$3.297\times10^{6}\begin{bmatrix} .65 & .081 & .75 & .15 & .0 & .325 & .1 & .094 \\ & .447 & .175 & .263 & .325 & .0 & .069 & .184 \\ & & 1.3 & -.488 & -.55 & .313 & .0 & .0 \\ & & & .894 & .338 & -.631 & .0 & .0 \\ & & & & 1.3 & -.163 & -.75 & .15 \\ & & & & & .894 & .175 & -.263 \\ & & & & & & .65 & .244 \\ & & & & & & & .447 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} R_{x1} \\ R_{y1} \\ 0 \\ -50{,}000 \\ 50{,}000 \\ 0 \\ R_{x4} \\ R_{y4} \end{Bmatrix}$$

$$3.297\times10^{6}\begin{bmatrix} 1.3 & -.488 & -.55 & .313 \\ & .894 & .338 & -.631 \\ & & 1.3 & -.163 \\ & & & .894 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -50{,}000 \\ 50{,}000 \\ 0 \end{Bmatrix}$$

$$\begin{aligned} u_2 &= -2.147\times10^{-3} \\ v_2 &= -4.455\times10^{-2} \\ u_3 &= 1.891\times10^{-2} \\ v_3 &= -2.727\times10^{-2} \end{aligned}$$

Example 6.2 cont.

- Element Results
  - Element 1

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{100}\begin{bmatrix} -10 & 0 & 15 & 0 & -5 & 0 \\ 0 & 0 & 0 & -10 & 0 & 10 \\ 0 & -10 & -10 & 15 & 10 & -5 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -2.147\times10^{-3} \\ -4.455\times10^{-2} \\ 1.891\times10^{-2} \\ -2.727\times10^{-2} \end{Bmatrix} = \begin{Bmatrix} -1.268\times10^{-3} \\ 1.727\times10^{-3} \\ -3.212\times10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = 3.297\times10^{7}\begin{bmatrix} 1 & .3 & 0 \\ .3 & 1 & 0 \\ 0 & 0 & .35 \end{bmatrix}\begin{Bmatrix} -1.268\times10^{-3} \\ 1.727\times10^{-3} \\ -3.212\times10^{-3} \end{Bmatrix} = \begin{Bmatrix} -24,709 \\ 44,406 \\ -37,063 \end{Bmatrix}\text{psi}$$

Example 6.2 cont.

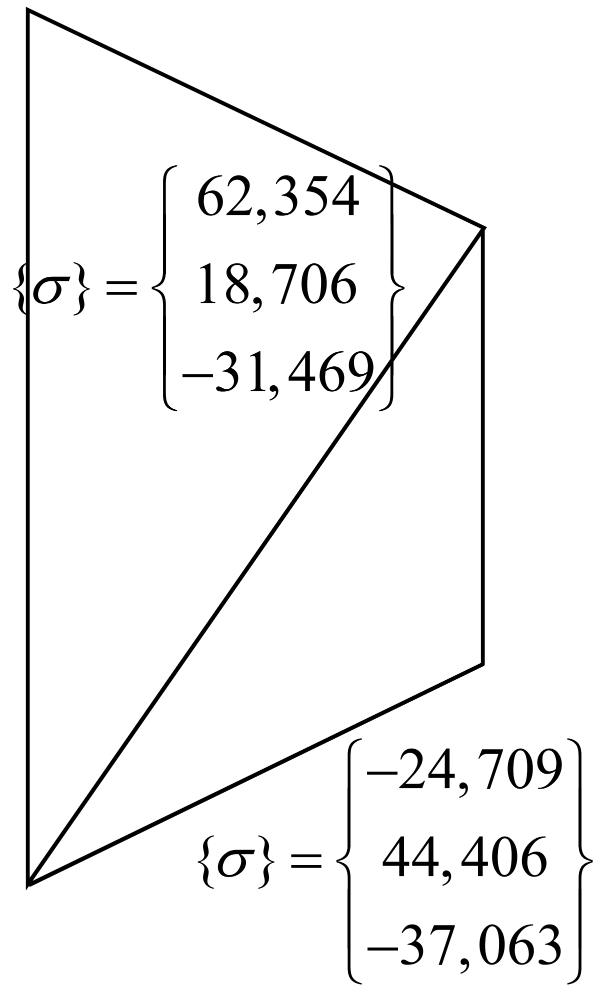
- Element Results
  - Element 2

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{200}\begin{bmatrix} -5 & 0 & 20 & 0 & -15 & 0 \\ 0 & -10 & 0 & 0 & 0 & 10 \\ -10 & -5 & 0 & 20 & 10 & -15 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 1.891\times 10^{-2} \\ -2.727\times 10^{-2} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.891\times 10^{-3} \\ 0 \\ -2.727\times 10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = 3.297\times 10^{7}\begin{bmatrix} 1 & .3 & 0 \\ .3 & 1 & 0 \\ 0 & 0 & .35 \end{bmatrix}\begin{Bmatrix} 1.891\times 10^{-3} \\ 0 \\ -2.727\times 10^{-3} \end{Bmatrix} = \begin{Bmatrix} 62,354 \\ 18,706 \\ -31,469 \end{Bmatrix}\text{psi}$$

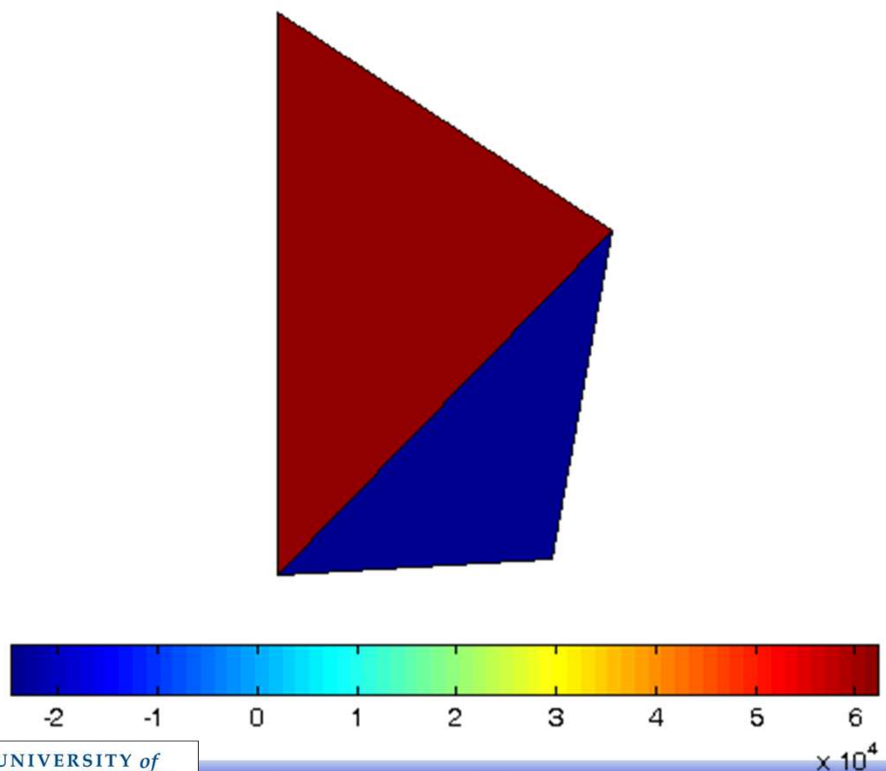
Example 6.2 cont.

- These stresses are constant over respective elements.
- large discontinuity in stresses across element boundaries

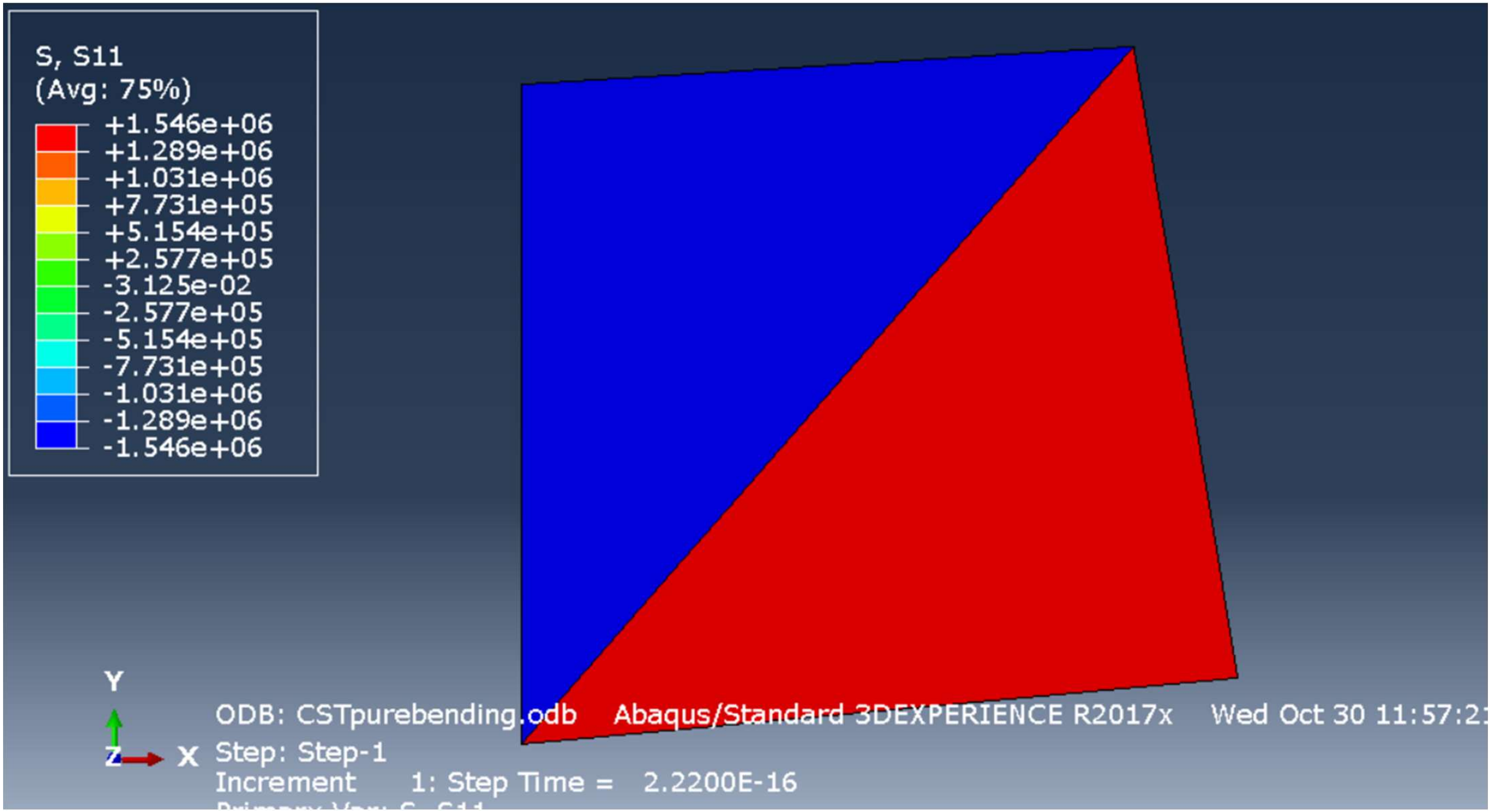


$$\{\sigma\} = \begin{Bmatrix} 62,354 \\ 18,706 \\ -31,469 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} -24,709 \\ 44,406 \\ -37,063 \end{Bmatrix}$$



Stress due to bending with CST elements



CST Element in Bending

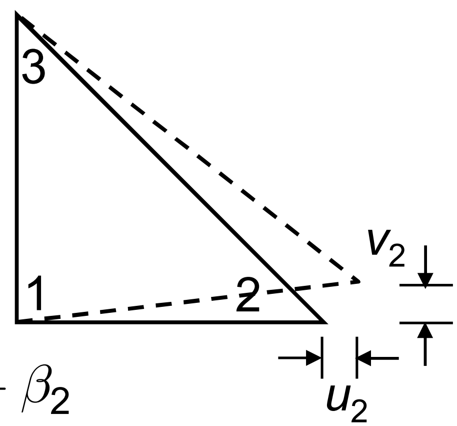
- Discussions
  - CST element performs well when strain gradient is small.
  - In pure bending problem, σ_{xx} in the neutral axis should be zero. Instead, CST elements show oscillating pattern of stress.
  - CST elements predict stress and deflection about ¼ of the exact values.
    - Strain along y-axis is supposed to be linear. But, CST elements can only have constant strain in y-direction.
    - CST elements also have spurious shear strain.

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v(x,y) = \beta_1 + \beta_2 x + \beta_3 y$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \alpha_2$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \beta_3$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \alpha_3 + \beta_2$$



How can we improve accuracy? What direction?

CST Element in Bending

- When u_2 & v_2 are not zero

$$u(x,y) = N_2(x,y)u_2 = \frac{u_2}{2A}(f_2 + b_2 x + c_2 y)$$

$$v(x,y) = N_2(x,y)v_2 = \frac{v_2}{2A}(f_2 + b_2 x + c_2 y)$$

$$b_2 = y_3 - y_1 = h$$
$$c_2 = x_1 - x_3 = 0$$

- Strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \alpha_2 = \frac{b_2}{2A}u_2 = \frac{hu_2}{2A}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \beta_3 = \frac{c_2}{2A}v_2 = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{c_2}{2A}u_2 + \frac{b_2}{2A}v_2 = \frac{hv_2}{2A}$$

$\Rightarrow$

Stress

$$\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx} = \frac{Ehu_2}{2A(1-\nu^2)}$$

$$\sigma_{yy} = \frac{\nu E}{1-\nu^2}\varepsilon_{xx} = \frac{\nu Ehu_2}{2A(1-\nu^2)}$$

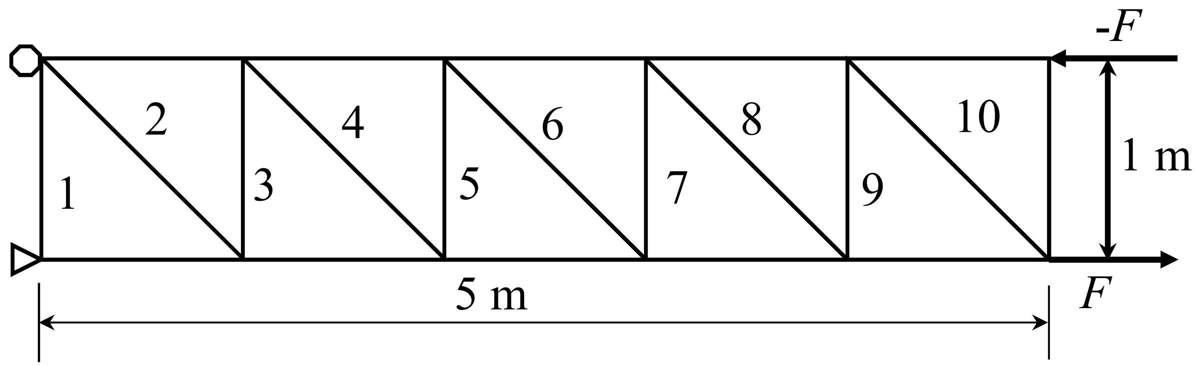
$$\tau_{xy} = G\gamma_{xy} = \frac{Ehv_2}{4A(1+\nu)}$$

Performance of CST element in bending

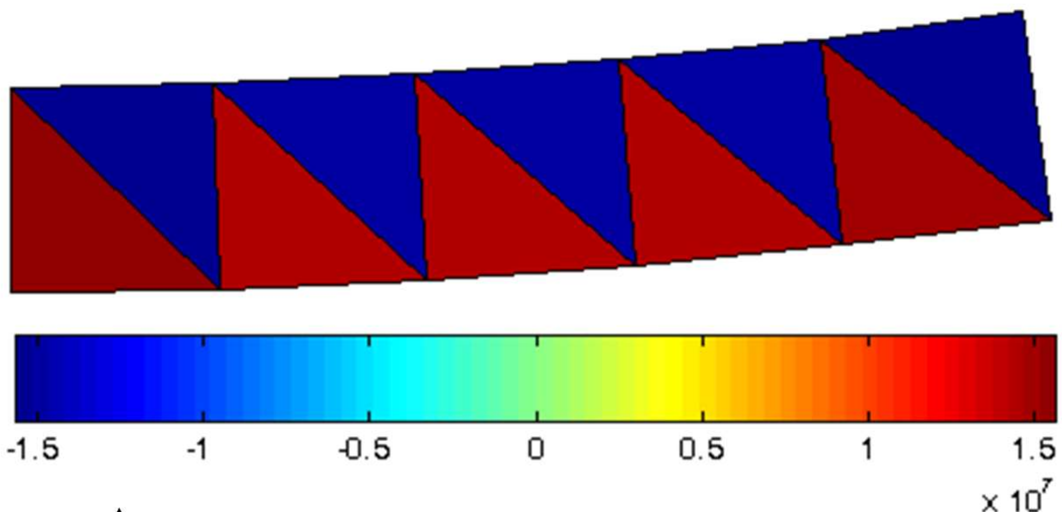
- Apply couple of 100 kN.m
- Thickness = 0.01m
- σ_{xx} is constant along the x-axis and linear along y-axis

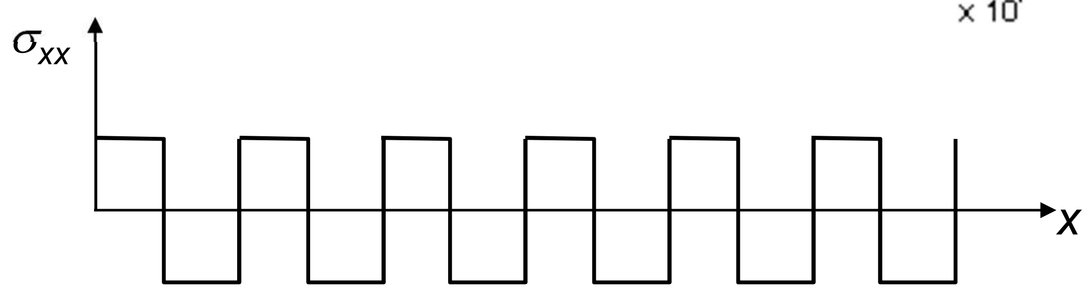
$$\sigma_{xx} = -\frac{My}{I}$$

- Exact Solution: σ_{xx} = 60 MPa
- Max deflection v_{max} = 0.0075 m



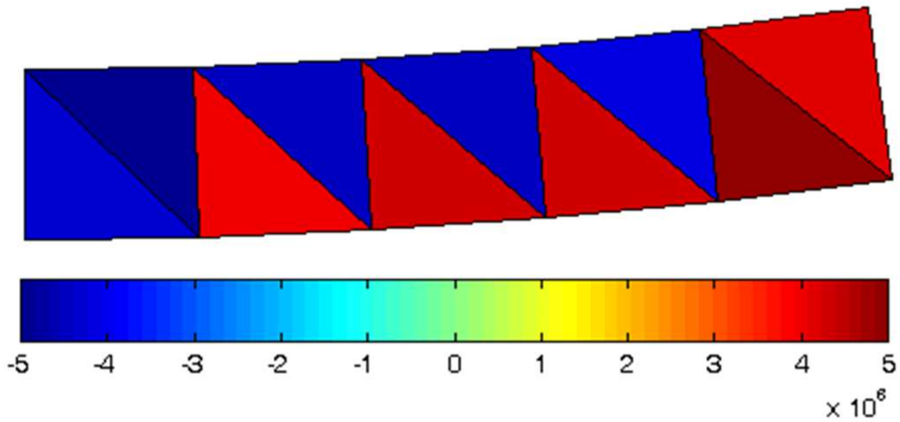
Max v = 0.0018



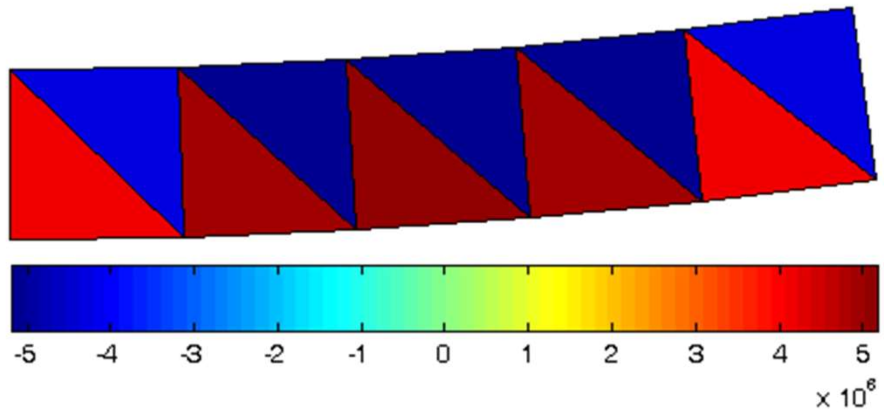


Performance of CST element in bending cont.

- y-normal stress and shear stress are supposed to be zero.



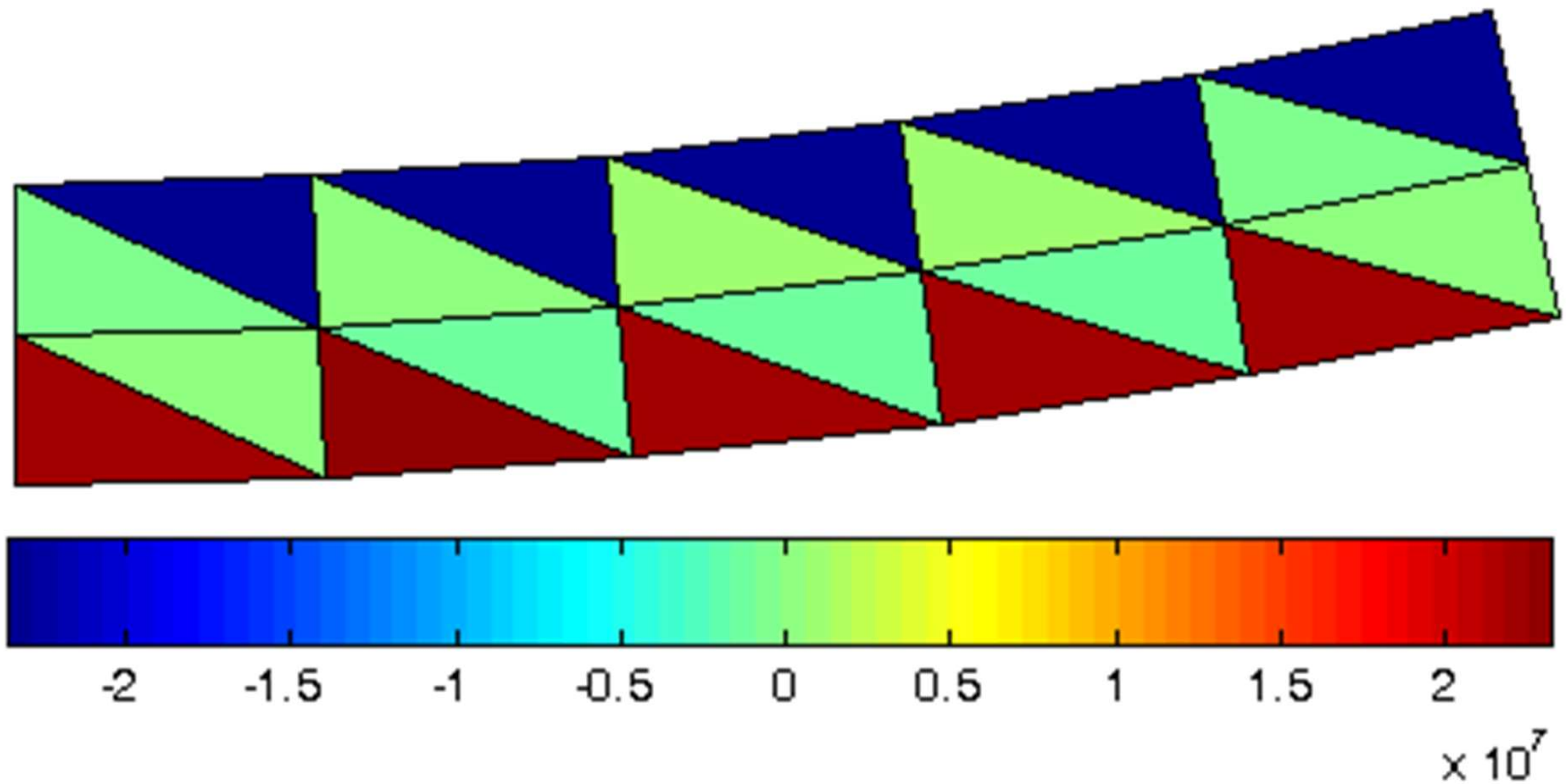
σ_{yy} Plot



τ_{xy} Plot

Performance of CST element in bending cont.

- Two-Layer Model
  - $\sigma_{xx} = 2.32 \times 10^7$
  - $v_{max} = 0.0028$



Plane solids: Bilinear Rectangular Element

Shape functions of rectangular element

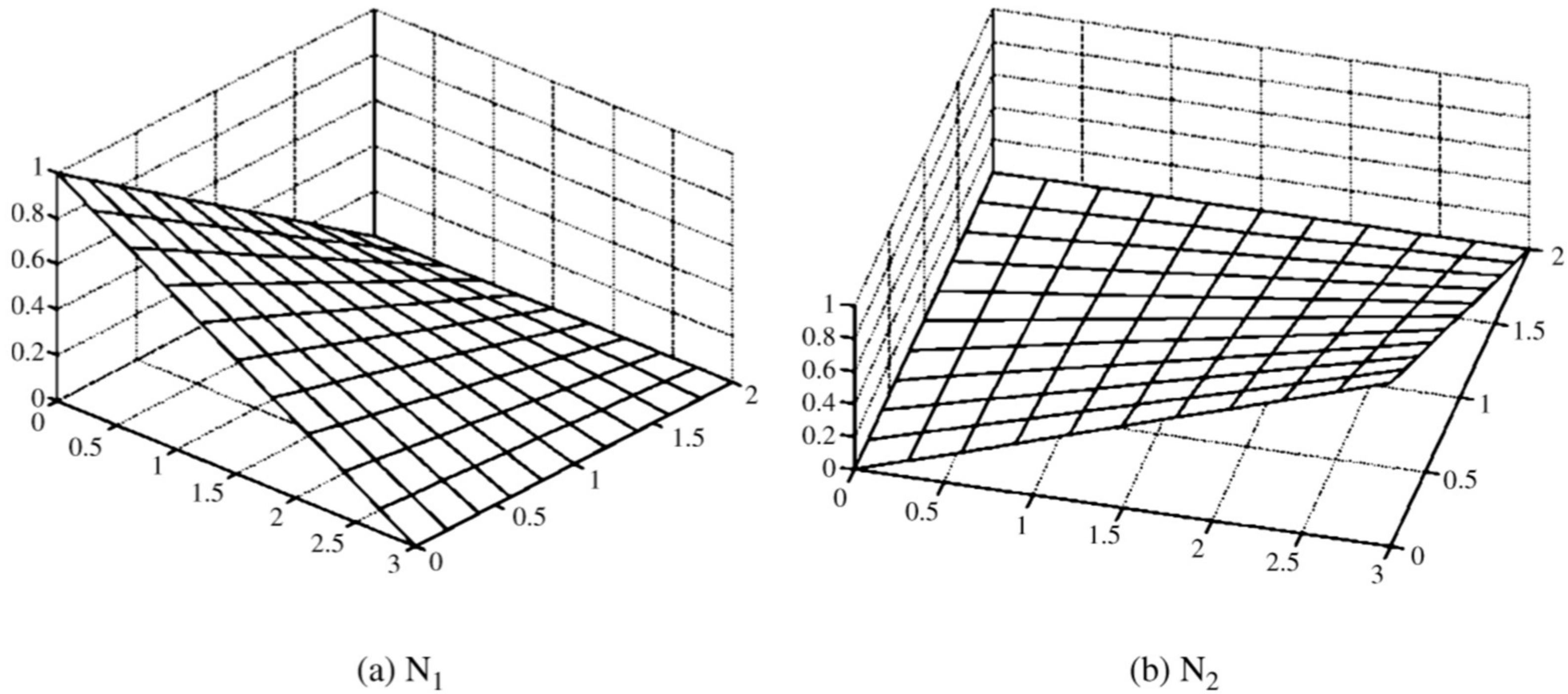


Figure 6.11 Three-dimensional surface plots of shape functions for a rectangular element

Stiffness matrix for square element

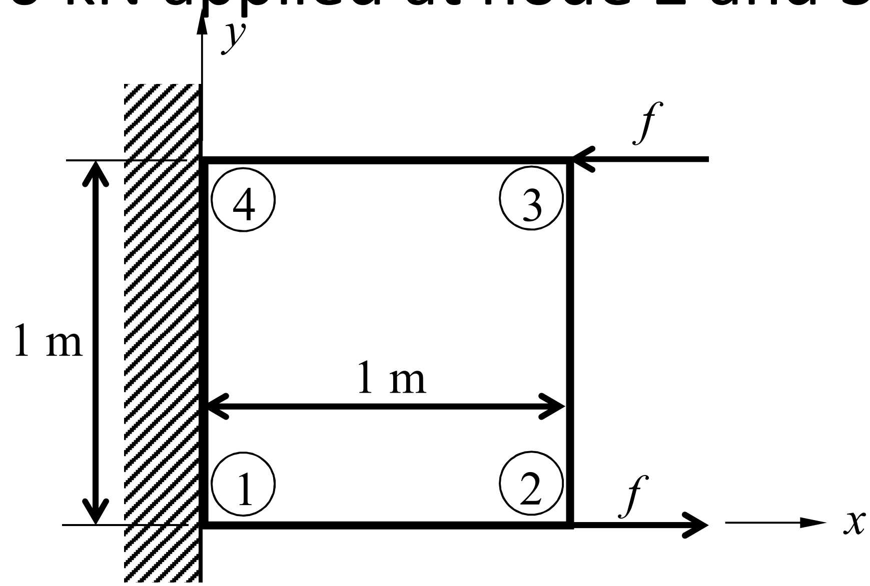
$$
[\mathbf{k}^{(e)}] = \frac{Eh}{1-\nu^2} \times \begin{bmatrix}
\frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} \\
\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} \\
-\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} \\
\frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} \\
\frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} \\
-\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} \\
\frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} \\
\frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}
$$

Ex) Bending of rectangular element

Equal and opposite force f = 100 kN applied at node 2 and 3

Thickness h = 0.1 in
E = 30×10^6 psi
v = 0.25

Calculate the stress and strain



Exact stress

$$\sigma_{xx} = -\frac{My}{I} = 6 \text{ MPa}$$

Stiffness matrix for square element

$$
[\mathbf{k}^{(e)}] = \frac{Eh}{1-\nu^2} \times \begin{bmatrix}
\frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} \\
\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} \\
-\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} \\
\frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} \\
\frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} \\
-\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} \\
\frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} \\
\frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}
$$

Ex) Bending of rectangular element cont.

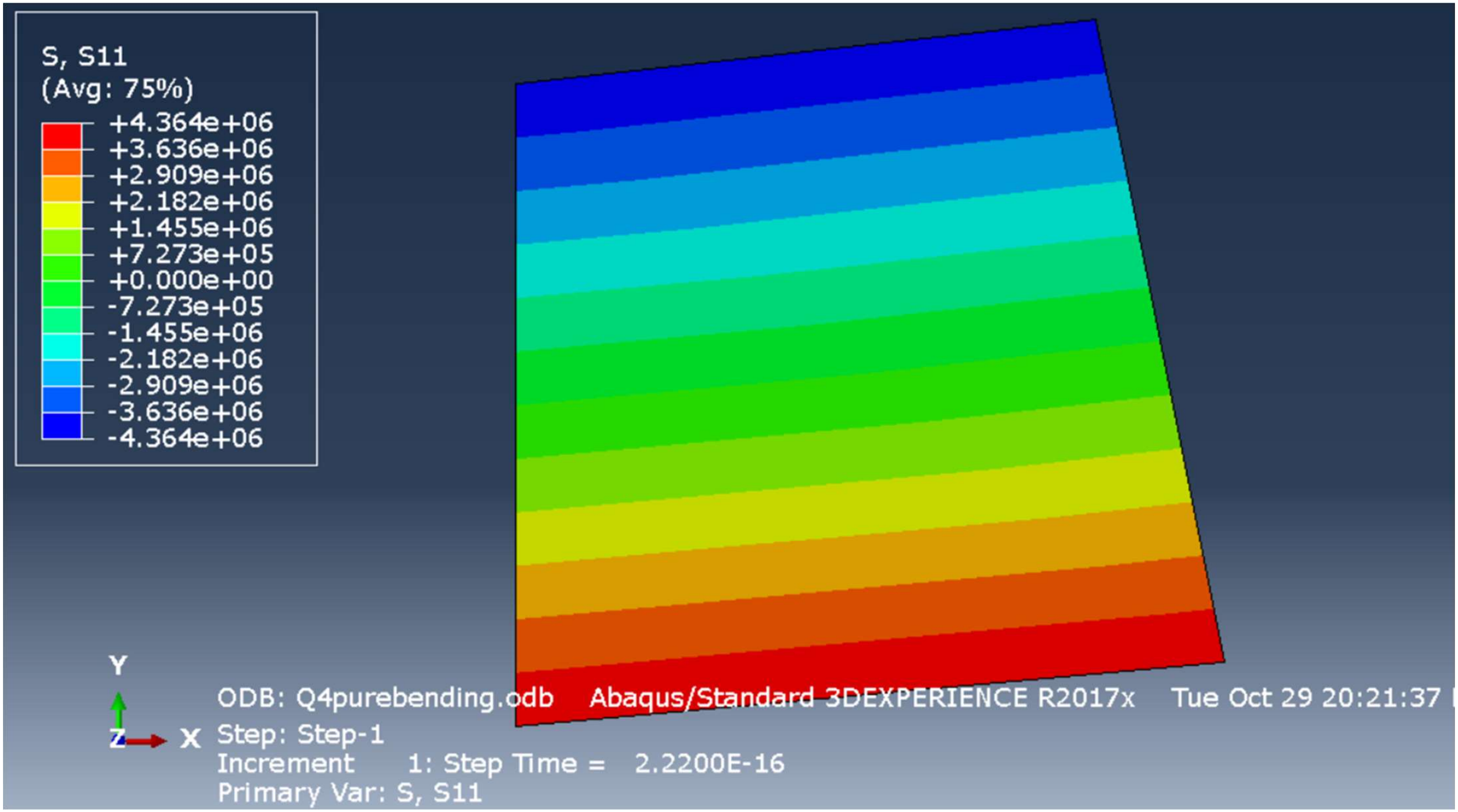
$$10^8\begin{bmatrix}4.89 & -1.67 & 0.44 & -0.33\\ -1.67 & 4.89 & 0.33 & -2.89\\ 0.44 & 0.33 & 4.89 & 1.67\\ -0.33 & -2.89 & 1.67 & 4.89\end{bmatrix}\begin{Bmatrix}u_2\\ v_2\\ u_3\\ v_3\end{Bmatrix}=\begin{Bmatrix}100{,}000\\ 0\\ -100{,}000\\ 0\end{Bmatrix}$$

$$u_2=0.4091\,\text{mm},\quad v_2=0.4091\,\text{mm}\quad u_3=-0.4091\,\text{mm},\quad v_3=0.4091\,\text{mm}$$

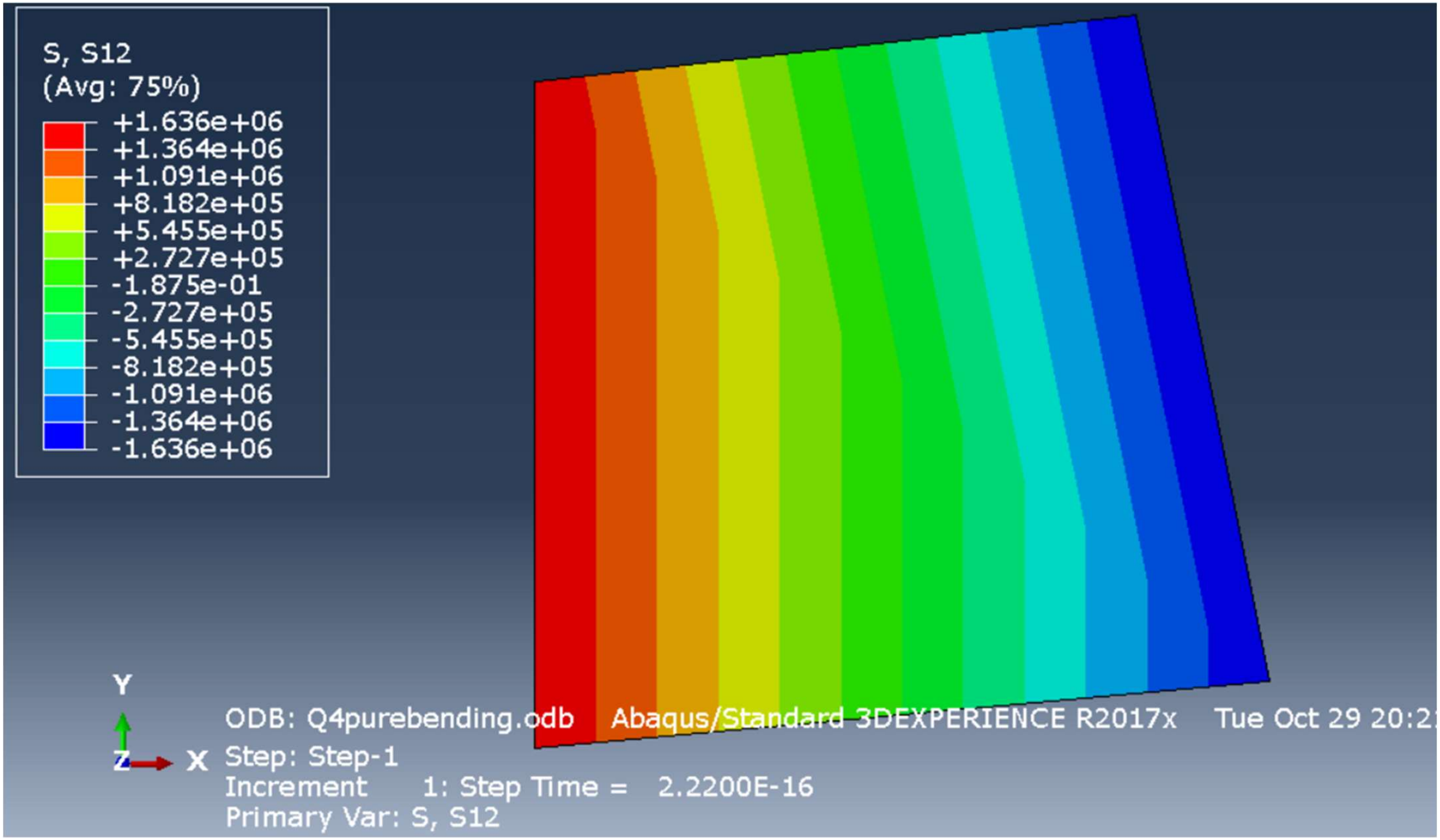
$$\{\varepsilon\}=\begin{bmatrix}y-1 & 0 & 1-y & 0 & y & 0 & -y & 0\\ 0 & x-1 & 0 & -x & 0 & x & 0 & 1-x\\ x-1 & y-1 & -x & 1-y & x & y & 1-x & -y\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 0.4091\\ 0.4091\\ -0.4091\\ 0.4091\\ 0\\ 0\end{Bmatrix}\times10^{-3}=\begin{Bmatrix}0.4091\times10^{-3}(1-2y)\\ 0\\ 0.4091\times10^{-3}(1-2x)\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}=\frac{10^{10}}{1-0.25^2}\begin{bmatrix}1 & 0.25 & 0\\ 0.25 & 1 & 0\\ 0 & 0 & 0.375\end{bmatrix}\begin{Bmatrix}0.4091\times10^{-3}(1-2y)\\ 0\\ 0.4091\times10^{-3}(1-2x)\end{Bmatrix}=\begin{Bmatrix}4.364(1-2y)\\ 1.091(1-2y)\\ 1.636(1-2x)\end{Bmatrix}\text{MPa}$$

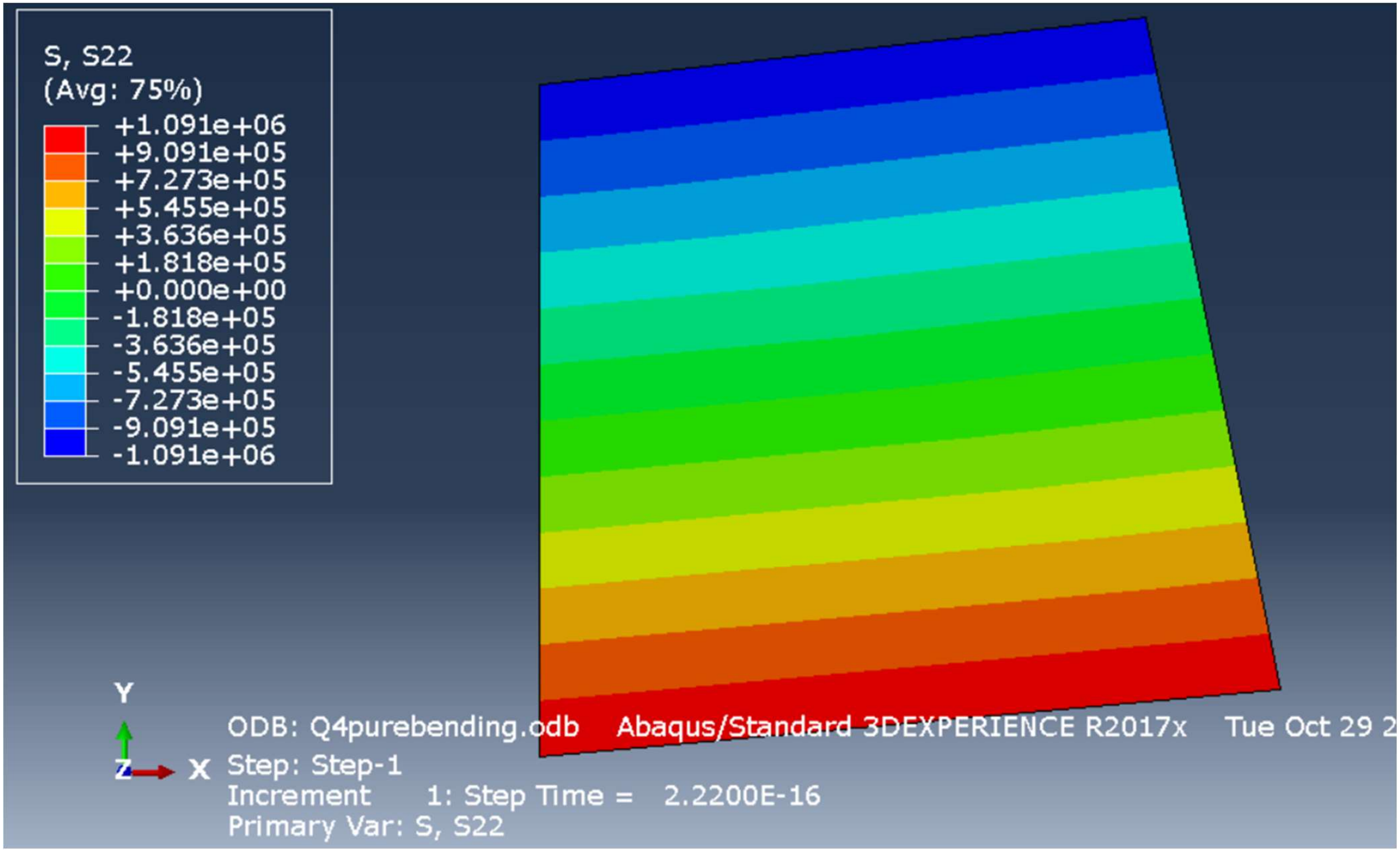
Ex) Bending of rectangular element cont.



Ex) Bending of rectangular element cont.

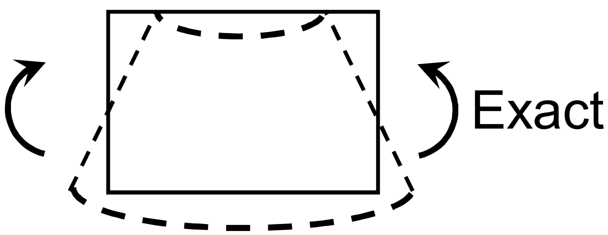


Ex) Bending of rectangular element cont.

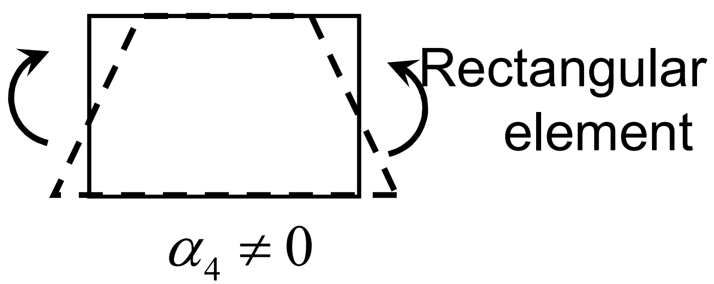


Ex) Bending of rectangular element cont.

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$
$$v = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$



$$\varepsilon_{xx} = \alpha_2 + \alpha_4 y$$
$$\varepsilon_{yy} = \beta_3 + \beta_4 x$$
$$\gamma_{xy} = (\alpha_3 + \beta_2) + \alpha_4 x + \beta_4 y$$

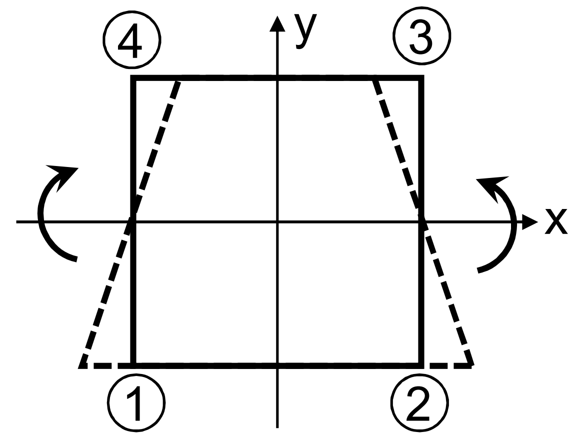


Rectangular Element in Bending

- When $-u_1 = u_2 = -u_3 = u_4 = \alpha_4$

$$u(x,y) = \sum_{i=1}^{4} N_i(x,y)u_i = -\alpha_4 xy$$

$$v(x,y) = \sum_{i=1}^{4} N_i(x,y)v_i = 0$$



- Strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -\alpha_4 y$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\alpha_4 x$$

$\Rightarrow$

Stress

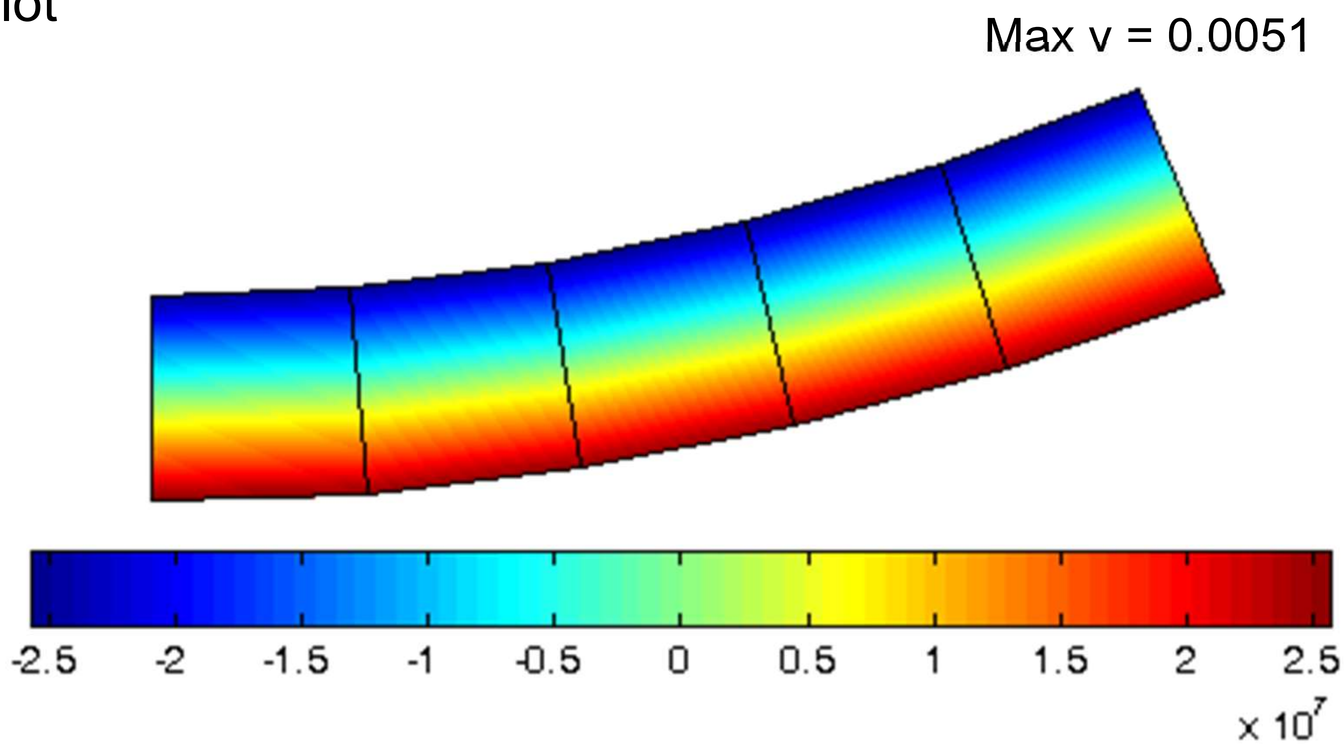
$$\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx} = -\frac{\alpha_4 E}{(1-\nu^2)} y$$

$$\sigma_{yy} = \frac{\nu E}{1-\nu^2}\varepsilon_{xx} = -\frac{\nu E \alpha_4}{(1-\nu^2)} y$$

$$\tau_{xy} = G\gamma_{xy} = -\frac{E\alpha_4}{2(1+\nu)} x$$

BEAM BENDING PROBLEM *cont.*

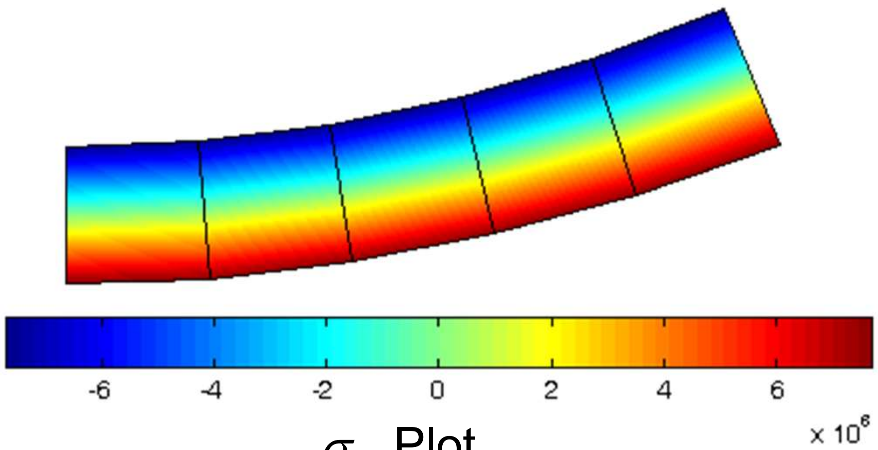
- Sxx Plot



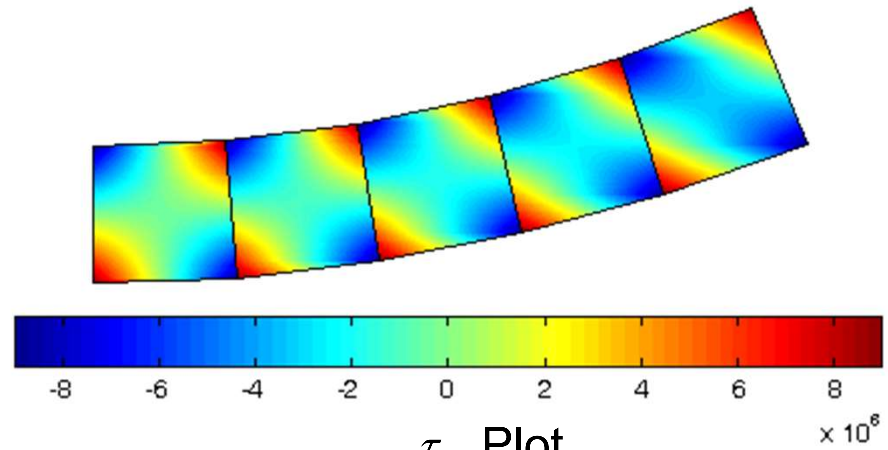
- Stress is constant along the x-axis (pure bending)
- linear through the height of the beam
- Deflection is much higher than CST element. In fact, CST element is too stiff. However, stress is inaccurate.

RECTANGULAR ELEMENT

- y-normal stress and shear stress are supposed to be zero.



σ_{yy} Plot



τ_{xy} Plot

ε_{xx} is a linear function of y alone
ε_{yy} is a linear function of x alone
γ_{xy} is a linear function of x and y

$$\varepsilon_{xx} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} u_I \qquad \varepsilon_{yy} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} v_I$$

$$\partial N_1 / \partial x = (y-1) \qquad \partial N_1 / \partial y = (x-1)$$

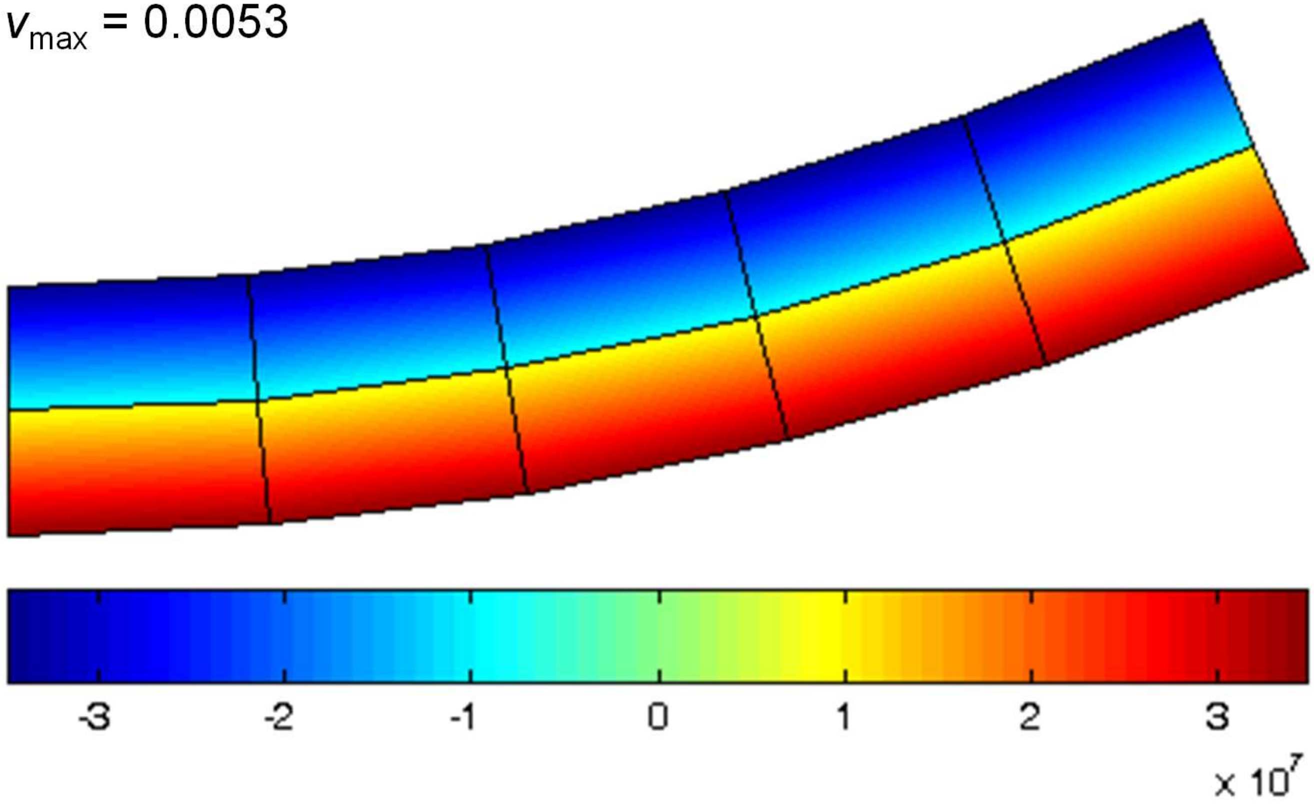
$$\partial N_2 / \partial x = -(y-1) \qquad \partial N_2 / \partial y = -x$$

$$\partial N_3 / \partial x = y \qquad \partial N_3 / \partial y = x$$

$$\partial N_4 / \partial x = -y \qquad \partial N_4 / \partial y = -(x-1)$$

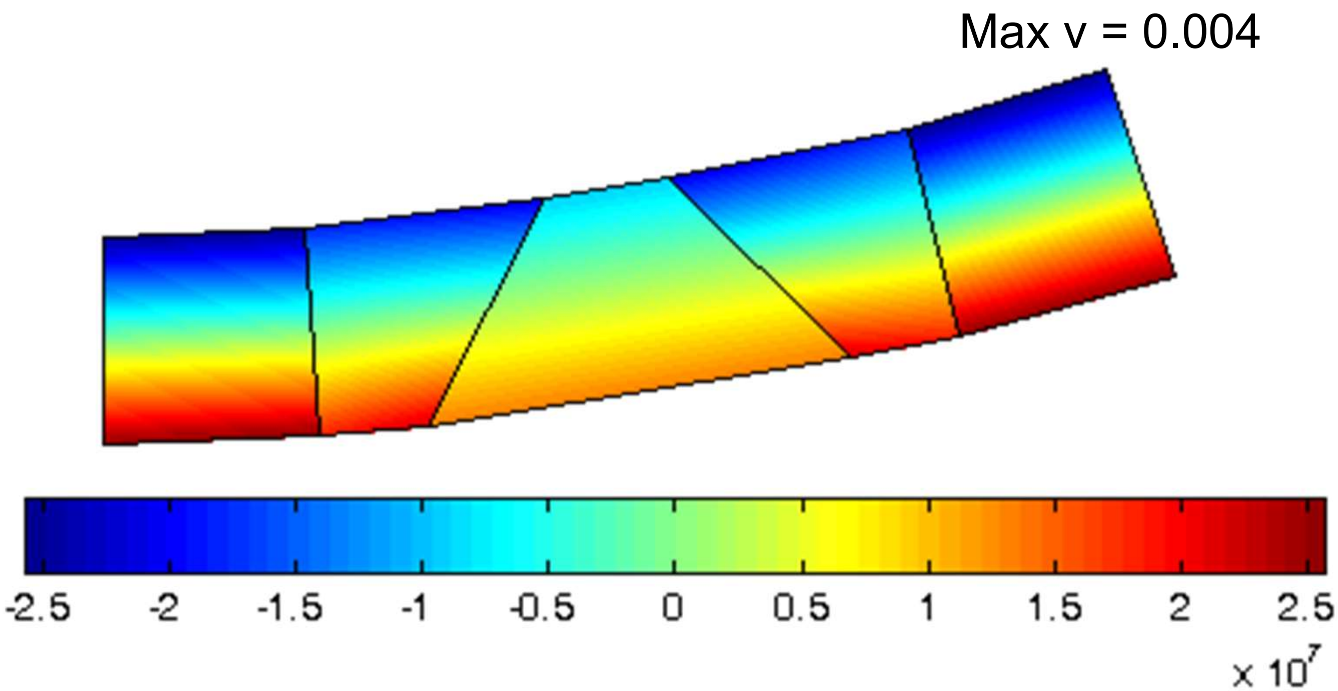
RECTANGULAR ELEMENT

- Two-Layer Model
  - $\sigma_{xx} = 3.48\times10^7$
  - $v_{max} = 0.0053$



BEAM BENDING PROBLEM *cont.*

- Distorted Element

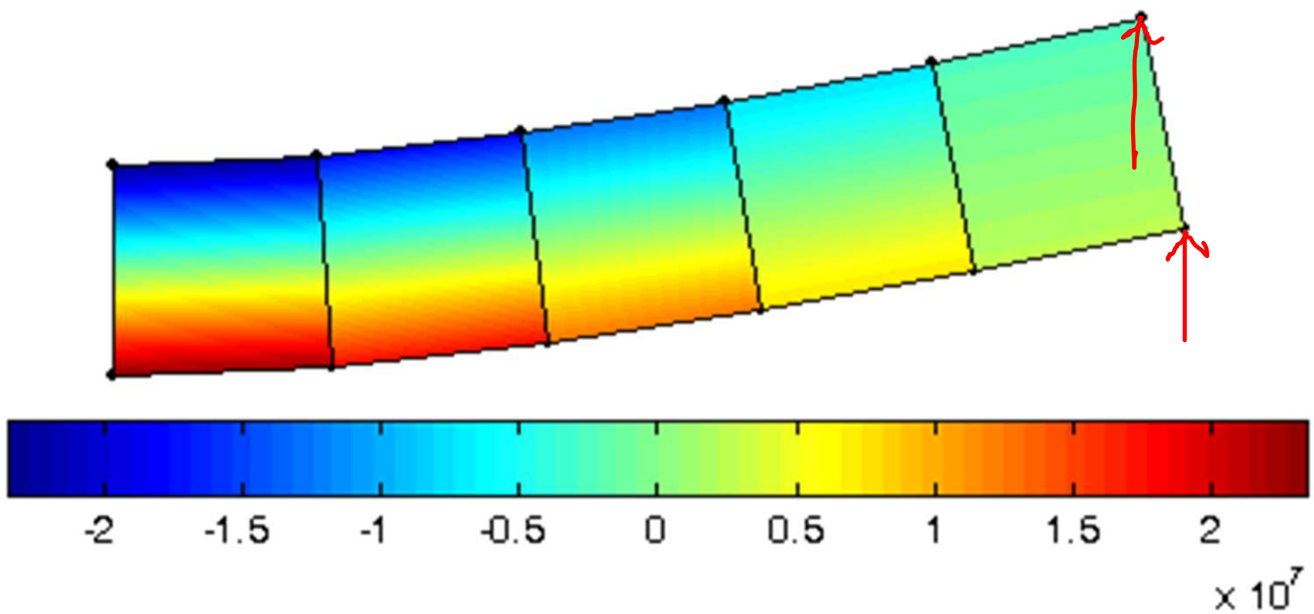


- As element is distorted, the solution is not accurate any more.

BEAM BENDING PROBLEM *cont.*

- Constant Shear Force Problem

Max v = 0.0035



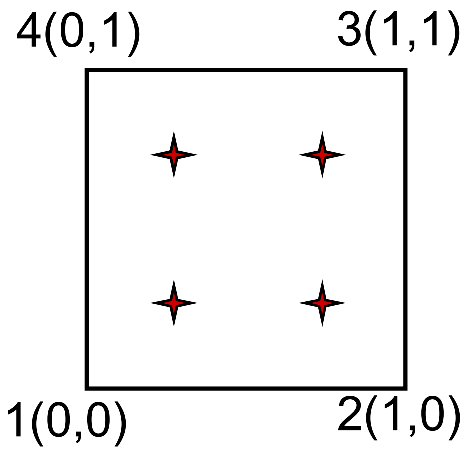
- Sxx is supposed to change linearly along x-axis. But, the element is unable to represent linear change of stress along x-axis. Why?
- Exact solution: v = 0.005 m and σ_{xx} = 6e7 Pa.

BEAM BENDING PROBLEM *cont.*

- Caution:
  - In numerical integration, we did not calculate stress at node points. Instead, we calculate stress at integration points.
  - Let's calculate stress at the bottom surface for element 1 in the beam bending problem.
  - Nodal Coordinates:1(0,0), 2(1,0), 3(1,1), 4(0,1)
  - Nodal Displacements:

    u = [0, 0.0002022, -0.0002022, 0]

    v = [0, 0.0002022, 0.0002022, 0]



  - Shape functions and derivatives

$$N_1 = (x-1)(y-1) \qquad \partial N_1/\partial x = (y-1) \qquad \partial N_1/\partial y = (x-1)$$

$$N_2 = -x(y-1) \qquad \partial N_2/\partial x = -(y-1) \qquad \partial N_2/\partial y = -x$$

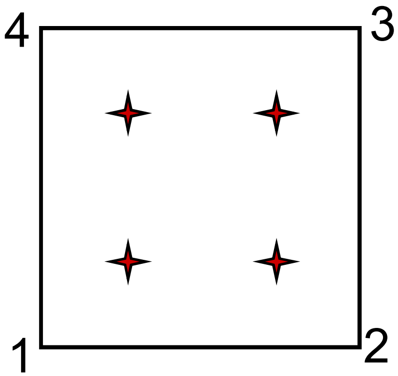
$$N_3 = xy \qquad \partial N_3/\partial x = y \qquad \partial N_3/\partial y = x$$

$$N_4 = -(x-1)y \qquad \partial N_4/\partial x = -y \qquad \partial N_4/\partial y = -(x-1)$$

BEAM BENDING PROBLEM *cont.*

- At bottom surface, y = 0

$$\partial N_1/\partial x = -1 \qquad \partial N_1/\partial y = x-1$$
$$\partial N_2/\partial x = 1 \qquad \partial N_2/\partial y = -x$$
$$\partial N_3/\partial x = 0 \qquad \partial N_3/\partial y = x$$
$$\partial N_4/\partial x = 0 \qquad \partial N_4/\partial y = -(x-1)$$



u = [0, 0.0002022, -0.0002022, 0]
v = [0, 0.0002022, 0.0002022, 0]

- Strain

$$\varepsilon_{xx} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} u_I = 1 \times 0.0002022$$

$$\varepsilon_{yy} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} v_I = -0.0002022 \times x + 0.0002022 \times x = 0$$

$$\gamma_{xy} = \sum_{I=1}^{4} \left(\frac{\partial N_I}{\partial x} v_I + \frac{\partial N_I}{\partial y} u_I \right) = 0.0002022 - 0.0004044x$$

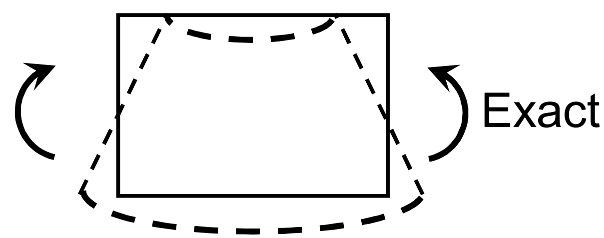
- Stress:

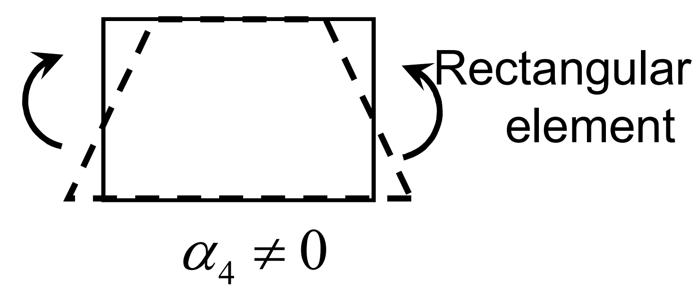
$$\{\sigma\} = [C]\{\varepsilon\} = \{4.44,\ 1.33,\ 1.55\} \times 10^7$$

- Discussions
  - Can't represent constant shear force problem because ε_{xx} must be a linear function of x.
  - Even if ε_{xx} can represent linear strain in y-direction, the rectangular element can't represent pure bending problem accurately.
  - Spurious shear strain makes the element too stiff.

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$
$$v = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$

$$\varepsilon_{xx} = \alpha_2 + \alpha_4 y$$
$$\varepsilon_{yy} = \beta_3 + \beta_4 x$$
$$\gamma_{xy} = (\alpha_3 + \beta_2) + \alpha_4 x + \beta_4 y$$



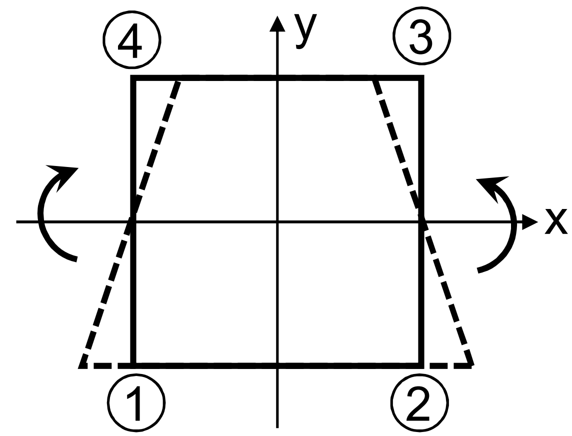


Rectangular Element in Bending

- When $-u_1 = u_2 = -u_3 = u_4 = \alpha_4$

$$u(x,y) = \sum_{i=1}^{4} N_i(x,y)u_i = -\alpha_4 xy$$

$$v(x,y) = \sum_{i=1}^{4} N_i(x,y)v_i = 0$$



- Strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -\alpha_4 y$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\alpha_4 x$$

⇨

Stress

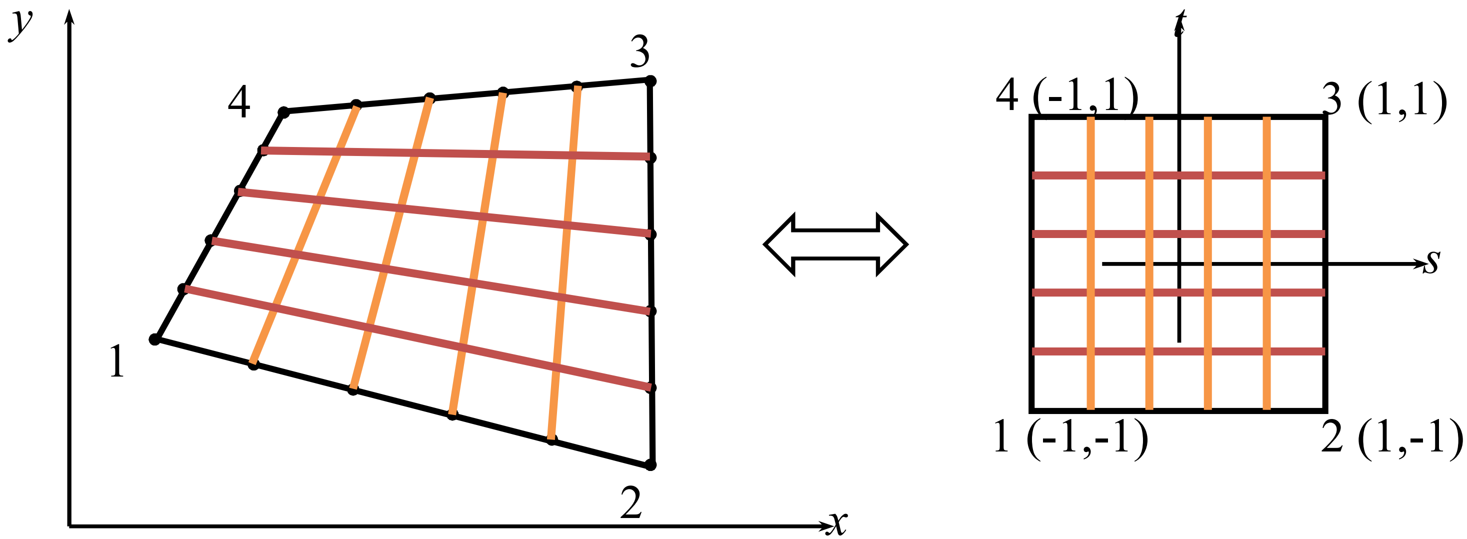
$$\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx} = -\frac{\alpha_4 E}{(1-\nu^2)}y$$

$$\sigma_{yy} = \frac{\nu E}{1-\nu^2}\varepsilon_{xx} = -\frac{\nu E \alpha_4}{(1-\nu^2)}y$$

$$\tau_{xy} = G\gamma_{xy} = -\frac{E\alpha_4}{2(1+\nu)}x$$

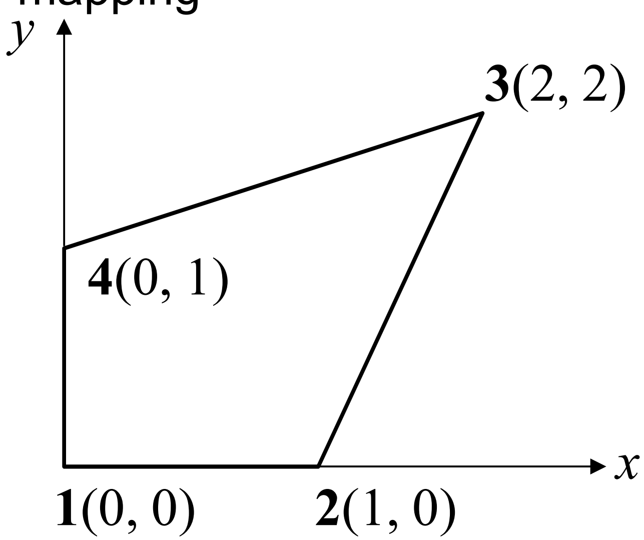
Isoperimetric Elements

Isoparametric Mapping



Jacobian of Mapping

Check validity of mapping



Coordinates mapping

$$x = \sum_{I=1}^{4} N_I x_I = N_2 + 2N_3 = \frac{1}{4}(3 + 3s + t + st)$$

$$y = \sum_{I=1}^{4} N_I y_I = 2N_3 + N_4 = \frac{1}{4}(3 + s + 3t + st)$$

Jacobian matrix

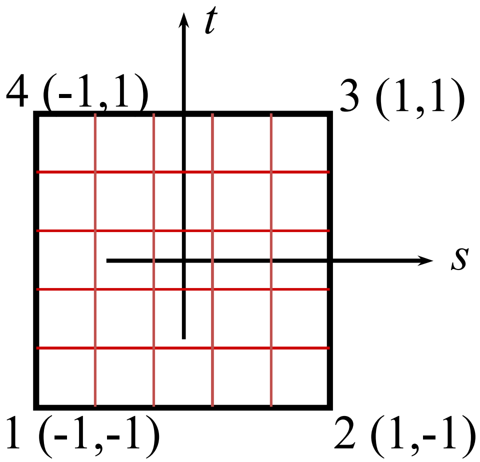
$$[\mathbf{J}] = \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3+t & 1+t \\ 1+s & 3+s \end{bmatrix}$$

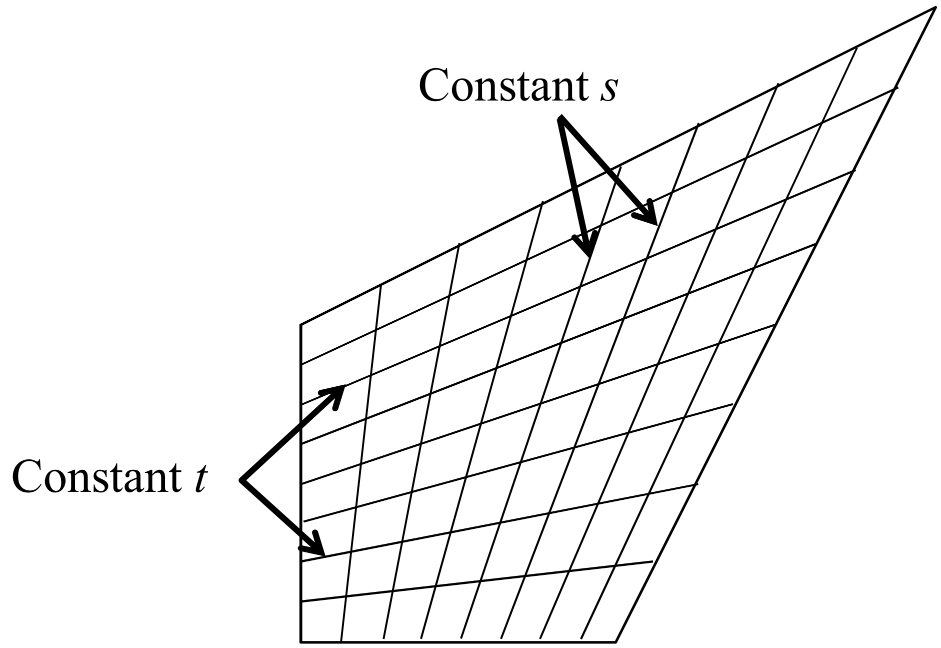
Jacobian

$$|\mathbf{J}| = \frac{1}{4}[(3+t)(3+s) - (1+t)(1+s)] = \frac{1}{2} + \frac{1}{8}s + \frac{1}{8}t$$

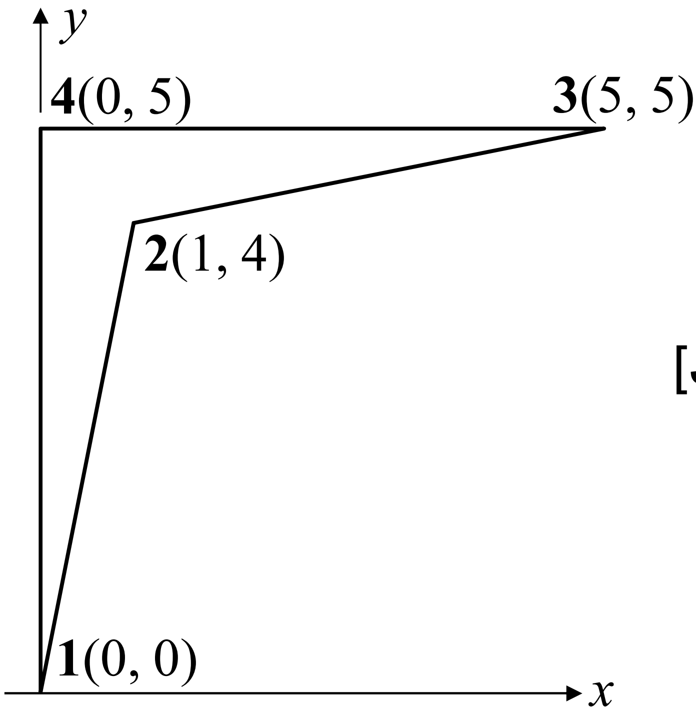
It is clear that |**J**| > 0 for –1 ≤ *s* ≤ 1 and –1 ≤ *t* ≤ 1.
Valid mapping

Jacobian of Mapping





Example Invalid Mapping



Coordinates mapping

$$x = \sum_{I=1}^{4} N_I x_I = \frac{1}{2}(1+s)(3+2t)$$

$$y = \sum_{I=1}^{4} N_I y_I = \frac{1}{2}(7+2s+3t-2st)$$

Jacobian matrix

$$[\mathbf{J}] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3+2t & 2-2t \\ 2+2s & 3-2s \end{bmatrix}$$

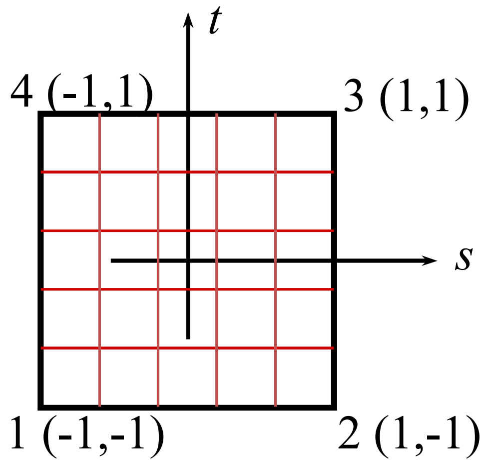
Jacobian

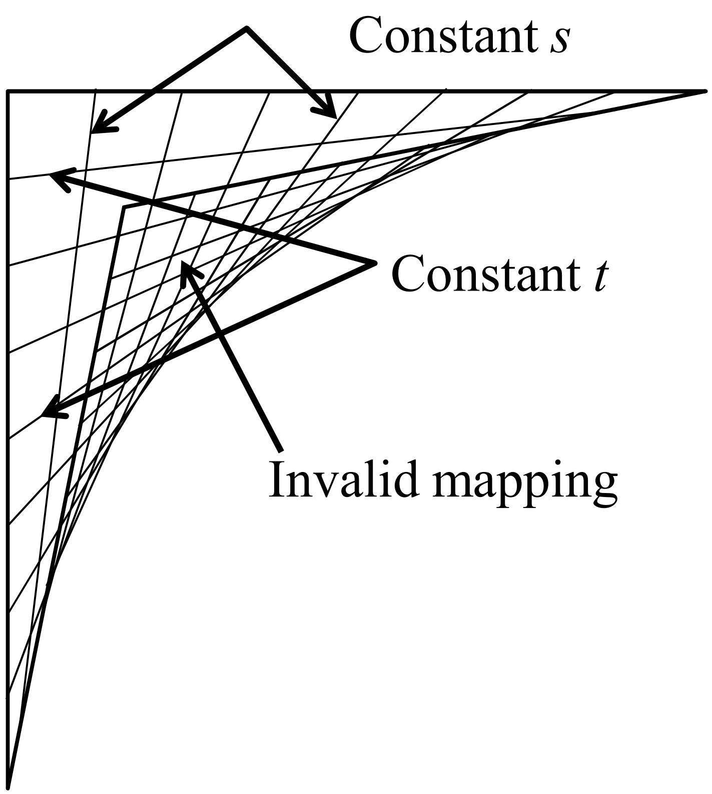
$$|\mathbf{J}| = \frac{1}{4}(5-10s+10t)$$

$|\mathbf{J}|$ = 0 at 5 – 10s + 10t = 0; i.e., s – t = ½

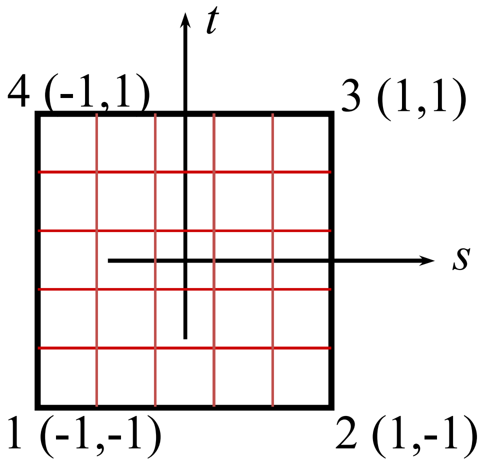
Invalid mapping

Example Invalid Mapping

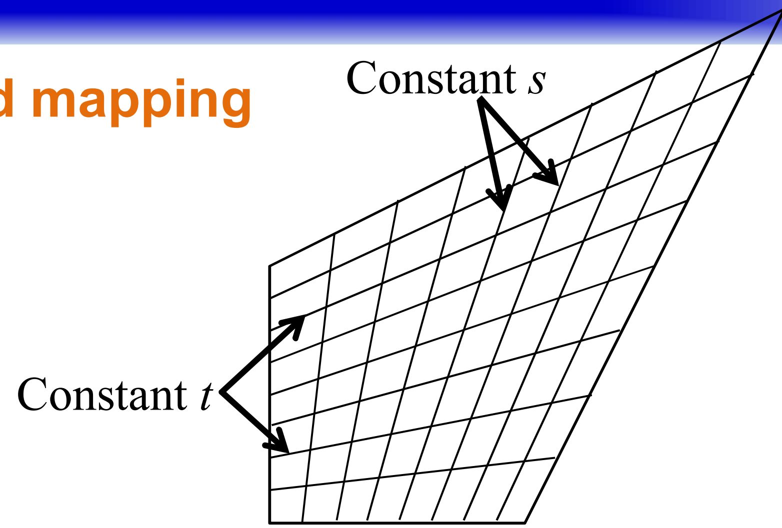




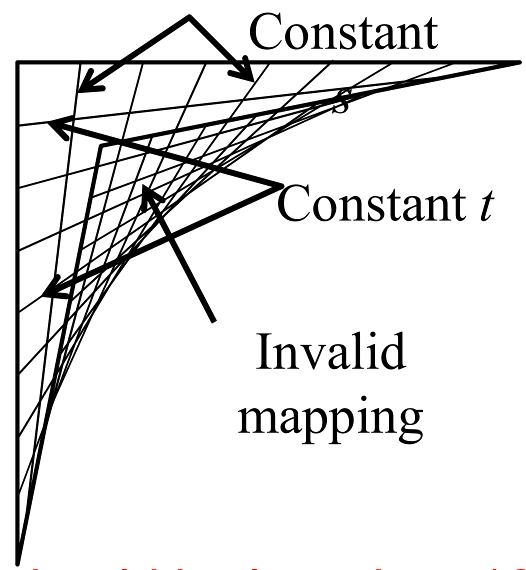
Jacobian of Mapping



Valid mapping



Invalid mapping

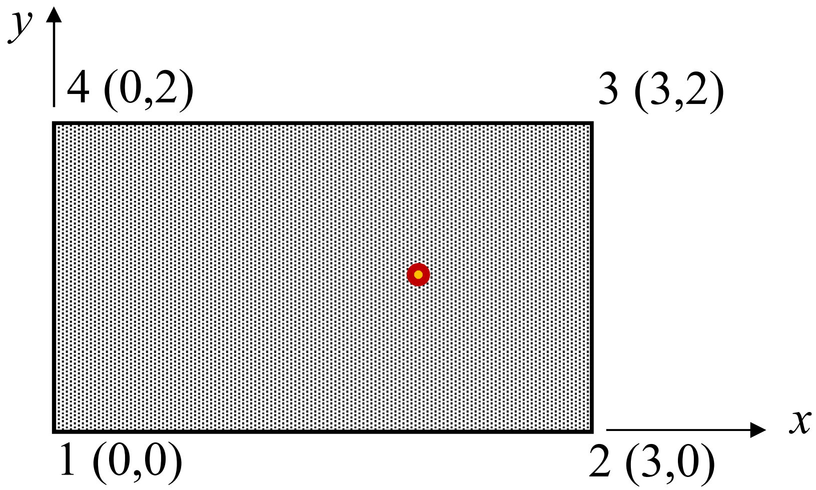


For valid mapping, all interior angles should be less than 180 degree

Example: Quadrilateral Isoparametric Element

$\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{0, 0, 1, 0, 2, 1, 0, 2\}$

Calculate strains at (x, y)=(2, 1)



Example *cont.*

1

$$\begin{cases} x = \sum_{I=1}^{4} N_I x_I = \frac{1}{4}(1+s)(1-t)\times 3 + \frac{1}{4}(1+s)(1+t)\times 3 = \frac{3}{2} + \frac{3}{2}s \\ y = \sum_{I=1}^{4} N_I y_I = \frac{1}{4}(1+s)(1+t)\times 2 + \frac{1}{4}(1-s)(1+t)\times 2 = 1+t \end{cases}$$

2

$$\begin{cases} x = \frac{3}{2} + \frac{3}{2}s = 2 \\ y = 1+t = 0 \end{cases} \Longrightarrow \begin{cases} s = \frac{1}{3} \\ t = 0 \end{cases}$$

3

$$[\mathbf{J}] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

Example *cont.*

4

$$\begin{Bmatrix} \dfrac{\partial N_I}{\partial x} \\ \dfrac{\partial N_I}{\partial y} \end{Bmatrix} = [\mathbf{J}]^{-1} \begin{Bmatrix} \dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix} = \begin{bmatrix} \dfrac{2}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix} = \begin{Bmatrix} \dfrac{2}{3}\dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix}$$

5

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} u_I = \sum_{I=1}^{4} \frac{2}{3}\frac{\partial N_I}{\partial s} u_I = \frac{2}{3}\left(\frac{1}{4}(1-t)\times 1 + \frac{1}{4}(1+t)\times 2\right) = \frac{1}{6}(3+t)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} v_I = \sum_{I=1}^{4} \frac{\partial N_I}{\partial t} v_I = \frac{1}{4}(1+s)\times 1 + \frac{1}{4}(1-s)\times 2 = \frac{1}{4}(3-s)$$

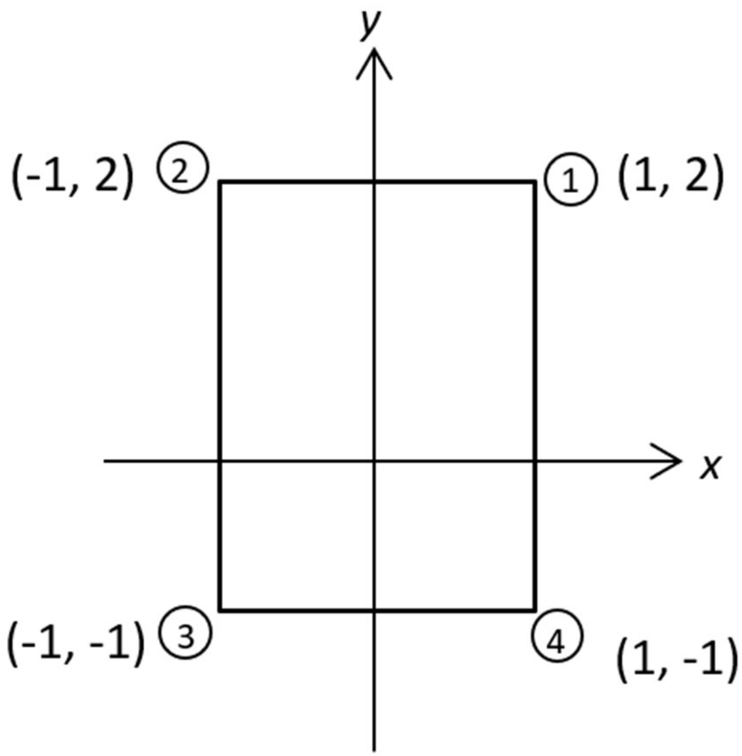
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \sum_{I=1}^{4}\left(\frac{\partial N_I}{\partial y} u_I + \frac{\partial N_I}{\partial x} v_I\right) = \sum_{I=1}^{4}\left(\frac{\partial N_I}{\partial t} u_I + \frac{2}{3}\frac{\partial N_I}{\partial s} v_I\right)$$

$$= \frac{1}{4}(-(1+s)\times 1 + (1+s)\times 2) + \frac{2}{3}\times\left(\frac{1}{4}(1+t)\times 1 - \frac{1}{4}(1+t)\times 2\right) = \frac{1}{4}(1+s) - \frac{1}{6}(1+t)$$

at s= 1/3, t=0 $\quad \varepsilon_{xx} = \dfrac{1}{2} \quad \varepsilon_{yy} = \dfrac{2}{3} \quad \gamma_{xy} = \dfrac{1}{6}$

Exercise

Write the Jacobian matrix for the rectangular element shown in the figure



$$N_1(s,t) = \frac{1}{4}(1-s)(1-t)$$

$$N_2(s,t) = \frac{1}{4}(1+s)(1-t)$$

$$N_3(s,t) = \frac{1}{4}(1+s)(1+t)$$

$$N_4(s,t) = \frac{1}{4}(1-s)(1+t)$$

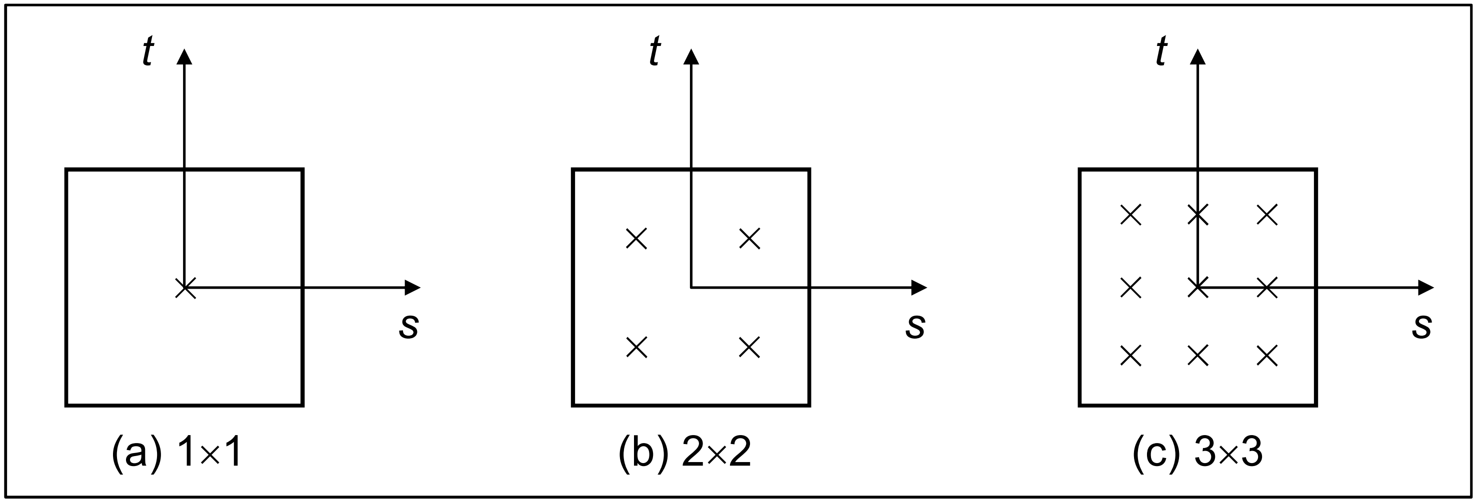
Numerical Integration

- Gauss Quadrature Points and Weights

n	Integration Points (s_i)	Weights (w_i)	Exact for polynomial of degree
1	0.0	2.0	1
2	±.5773502692	1.0	3
3	±.7745966692 0.0	.5555555556 .8888888889	5
4	±.8611363116 ±.3399810436	.3478546451 .6521451549	7
5	±.9061798459 ±.5384693101 0.0	.2369268851 .4786286705 .5688888889	9

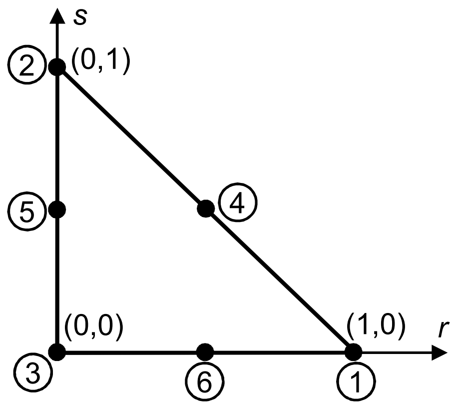
Numerical Integration

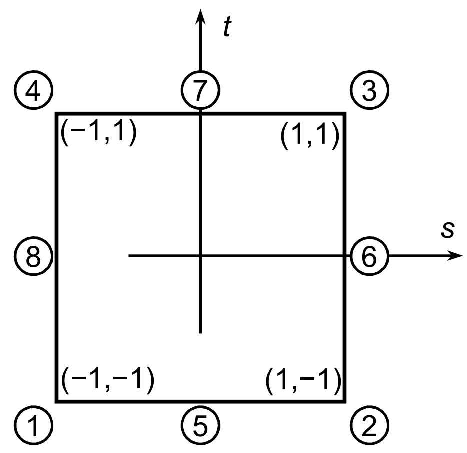
- 2D Gauss Quadrature



Exercise

- Write the interpolation basis functions (polynomials) of u(x,y) and v(x,y) for (a) LST element and (b) Q8 element

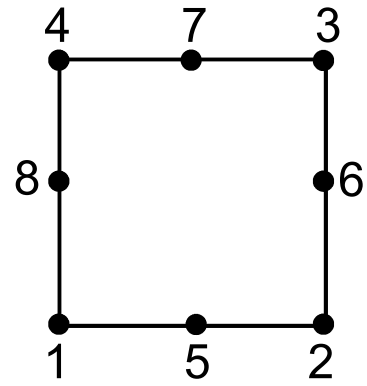




Higher-Order Element (Q8)

- Higher-Order Element?
  - 8-Node Rectangular Element

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy$$
$$+ a_5 y^2 + a_6 x^2 y + a_7 xy^2$$



  - Strain

$$\frac{\partial u(x,y)}{\partial x} = a_1 + 2a_3 x + a_4 y + 2a_6 xy + a_7 y^2$$

  - Can this element accurately represent pure bending and constant shear force problem?

1

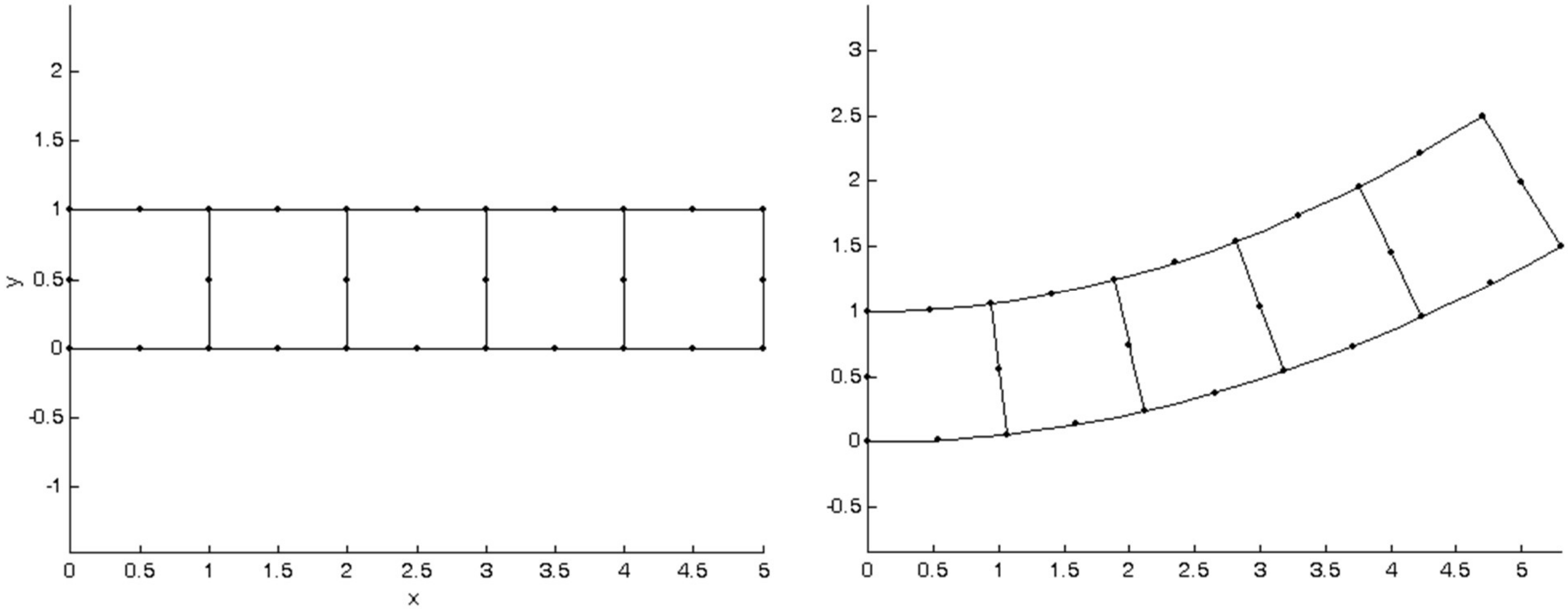
x y

x^2 xy y^2

x^3 x^2y xy^2 y^3

Beam Bending (Q8) *cont.*

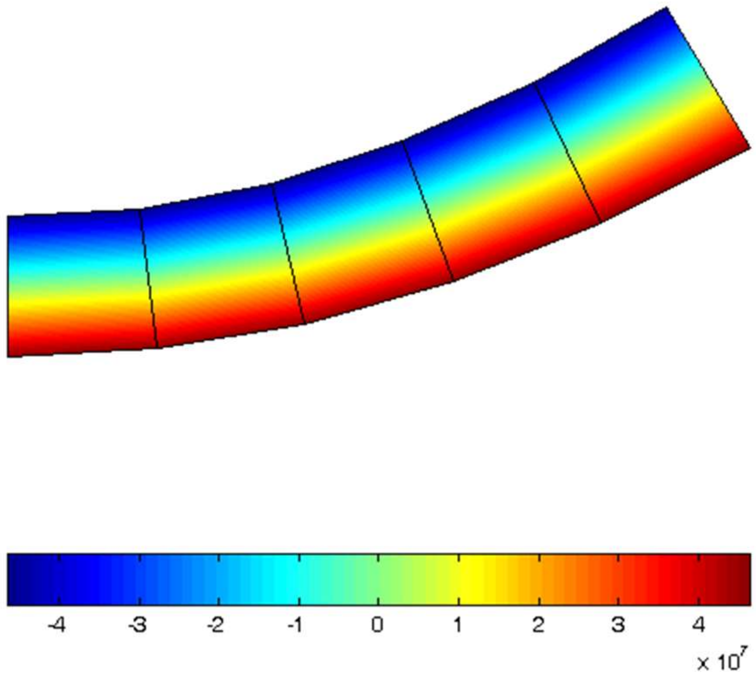
- 8-Node Rectangular Elements



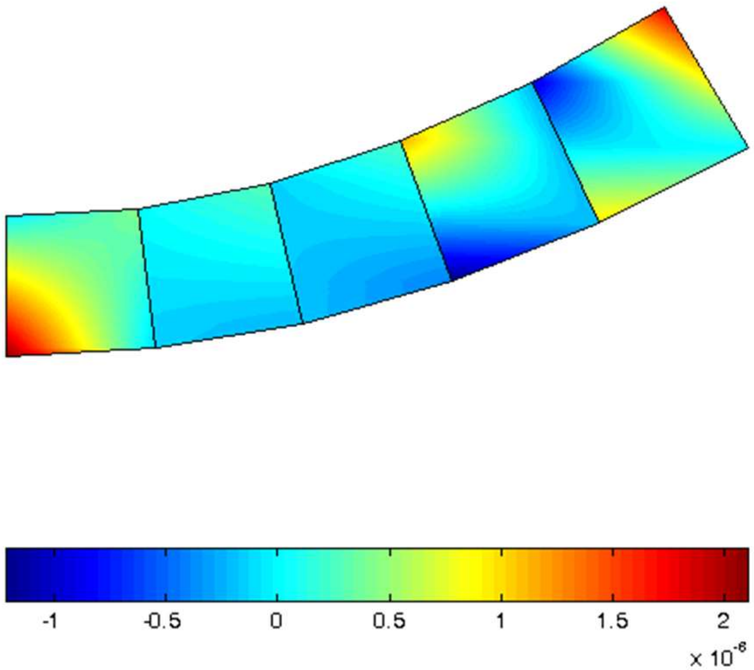
- Tip Displacement = 0.0075 m,   Exact!

Beam Bending (Q8) *cont.*

- If the stress at the bottom surface is calculated, it will be the exact stress value.



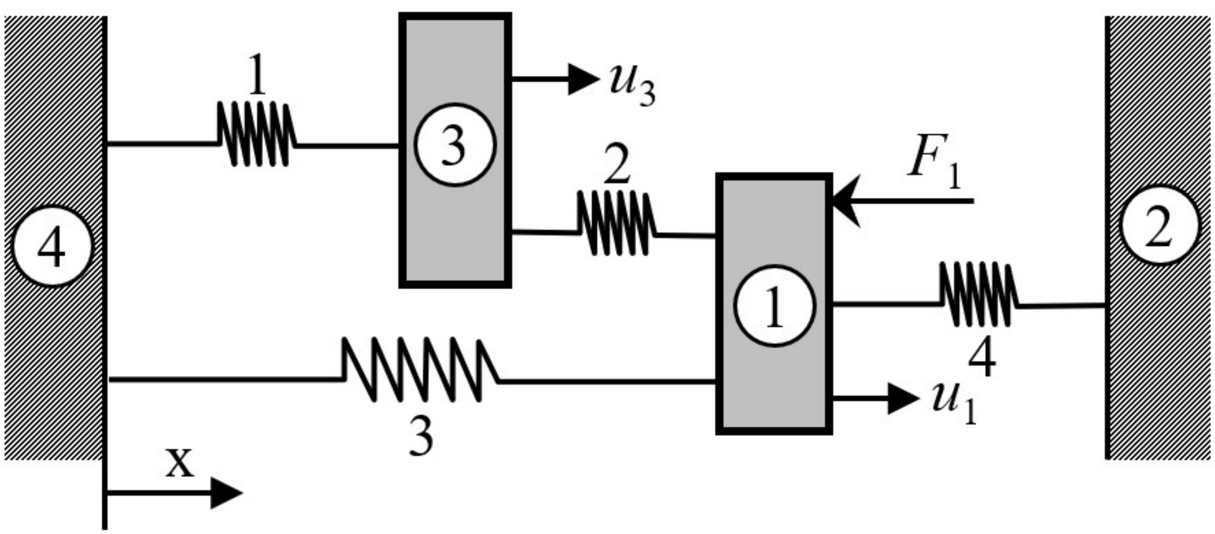
Sxx



Syy

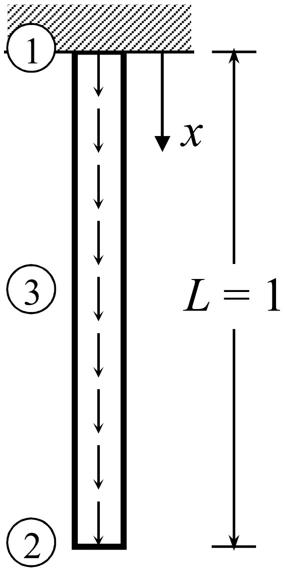
Quiz 1

- Four rigid bodies are connected by four springs as shown in the figure. A horizontal force of 1,000 N is applied on Body 1 as shown in the figure. The spring constants (N/mm) are $k^{(1)} = 500$, $k^{(2)} = 300$, $k^{(3)} = 400$, and $k^{(4)} = 300$. Do not change node and element numbers.
- Write the structural matrix equation with unknown reactions and known forces. Do not solve the matrix equation!



Quiz2

A vertical bar is under gravity, which is modeled by 1D 3-node bar element. Calculate displacement u(x) and strain ε_{xx}(x) when the nodal displacements are given.

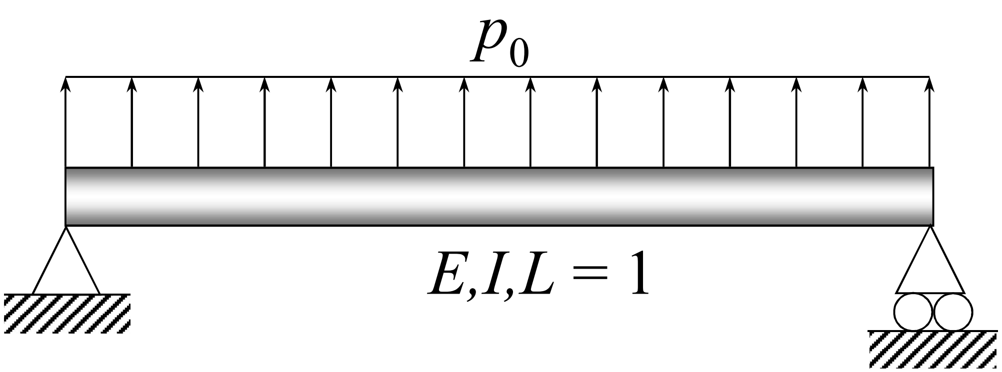


$$N_1(x) = 1 - 3x + 2x^2$$
$$N_2(x) = -x + 2x^2$$
$$N_3(x) = 4x - 4x^2$$

$$u_1 = 0, \quad u_2 = \frac{1}{2}, \quad u_3 = \frac{3}{8}$$

Quiz3

- Calculate the deflection curve v(s), bending moment M(s), and shear force Vy(s) of the simply-supported beam :

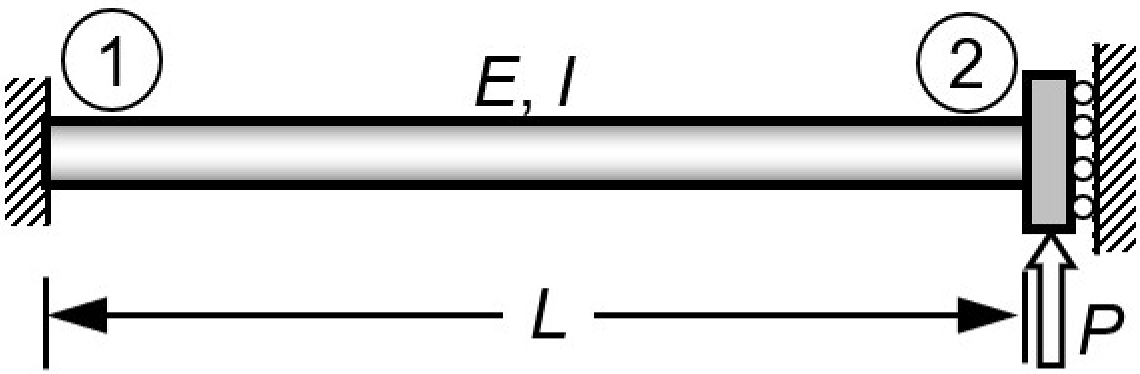


$$N_1(s) = 1 - 3s^2 + 2s^3$$
$$N_2(s) = L(s - 2s^2 + s^3)$$
$$N_3(s) = 3s^2 - 2s^3$$
$$N_4(s) = L(-s^2 + s^3)$$

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} pL/2 \\ pL^2/12 \\ pL/2 \\ -pL^2/12 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ C_1 \\ F_2 \\ C_2 \end{Bmatrix}$$

Quiz 3

- A beam is fixed on the left and not allowed to rotate at the right end. Calculate the deflection curve v(s)

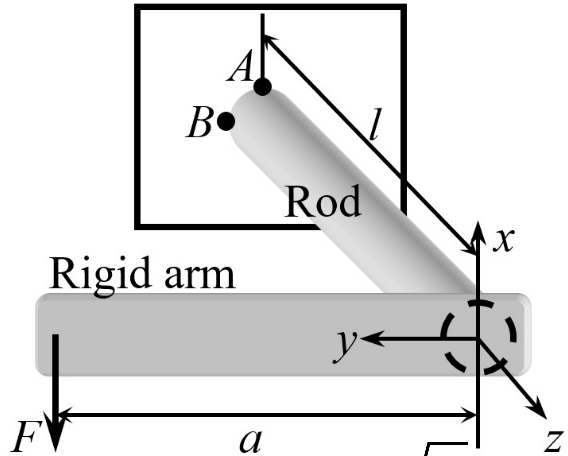


$$N_1(s) = 1 - 3s^2 + 2s^3$$
$$N_2(s) = L(s - 2s^2 + s^3)$$
$$N_3(s) = 3s^2 - 2s^3$$
$$N_4(s) = L(-s^2 + s^3)$$

$$[\mathbf{k}^{(e)}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Quiz 4

- The bracket consists of a rigid arm and a flexible rod. The latter has the following properties: moment of inertia $I = 1.0$, polar moment of inertia $J = 0.5$, radius $r = 0.1$, and length $l = 10.0$. When a vertical force $F = 1.0$ is applied, calculate the stress matrix at A.



- Bending stress: $\sigma_b = -\dfrac{My}{I}$
- Torsional shear stress: $\tau = \dfrac{Tr}{J}$
- Transverse shear stress: $\tau = \dfrac{VQ}{Ib}$