CHAP 3 FINITE ELEMENT ANALYSIS OF BEAMS AND FRAMES

3.1 ELEMENTARY BEAM THEORY

INTRODUCTION

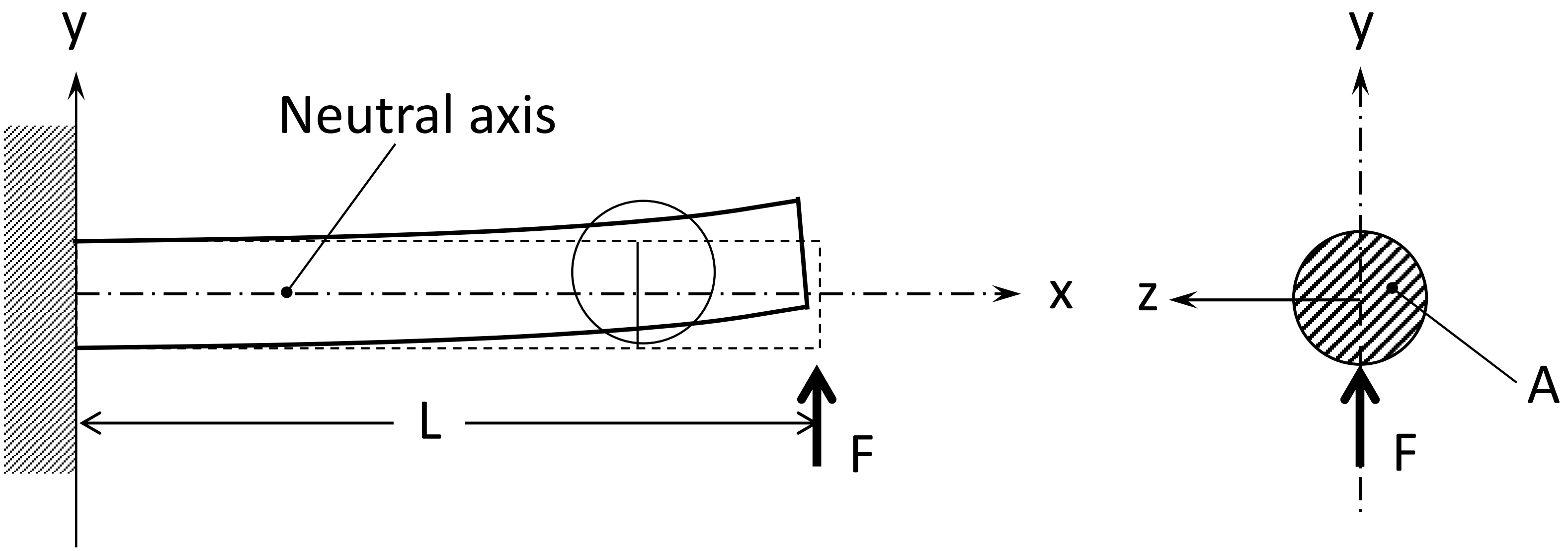
- We learned **Direct Stiffness Method** in Chapter 1
  - Limited to simple elements such as 1D bars
- In Chapter 3, **Galerkin Method** and **Principle of Minimum Potential Energy** can be applied to more complex elements
- we will learn **Energy Method** to build beam finite element
  - Structure is in equilibrium when the potential energy is minimum
- **Potential energy**: Sum of strain energy and potential of applied loads
- Interpolation scheme:

$$\Pi = \underbrace{U}_{\text{Strain energy}} + \overbrace{V}^{\text{Potential of applied loads}}$$

$$\underbrace{v(x)}_{\text{Beam deflection}} = \underbrace{\lfloor \mathbf{N}(x) \rfloor}_{\text{Interpolation function}} \cdot \underbrace{\{\mathbf{q}\}}_{\text{Nodal DOF}}$$

Review of Beam Theory

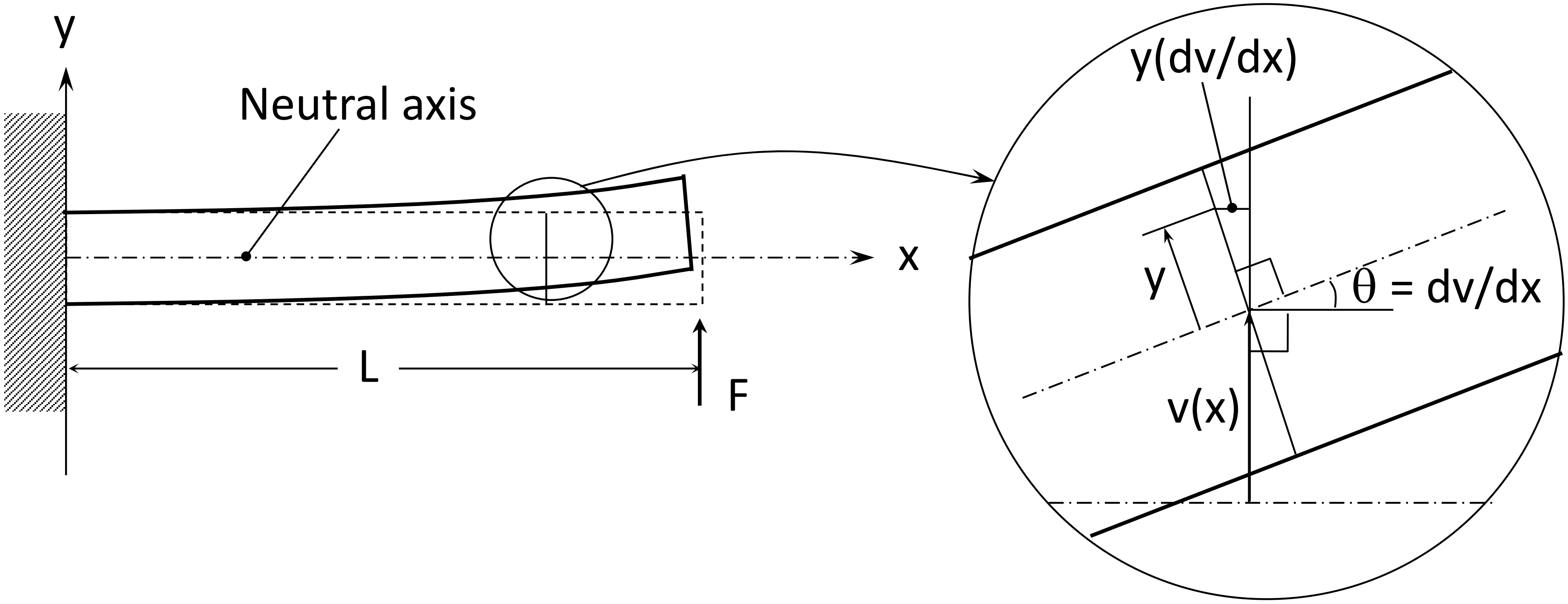
- Euler-Bernoulli Beam Theory
  - can carry the transverse load
  - slope can change along the span (x-axis)
  - Cross-section is symmetric w.r.t. xy-plane
  - The y-axis passes through the centroid
  - Loads are applied in xy-plane (plane of loading)



BEAM THEORY *cont.*

- Euler-Bernoulli Beam Theory cont.
  - Plane sections normal to the beam axis remain plane and normal to the axis after deformation (no shear stress)
  - Transverse deflection (deflection curve) is function of x only: **v(x)**
  - Displacement in x-dir is function of x and y: **u(x, y)**

$$u(x,y) = u_0(x) - y\frac{dv}{dx} \quad \Rightarrow \quad \varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du_0}{dx} - y\frac{d^2v}{dx^2} \qquad \theta = \frac{dv}{dx}$$



BEAM THEORY *cont.*

$$\boxed{\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\mathrm{d}u_0}{\mathrm{d}x} - y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}}$$

- Euler-Bernoulli Beam Theory cont.
  - Strain along the beam axis: $\varepsilon_0 = \mathrm{d}u_0 / \mathrm{d}x$
  - Strain ε_{xx} varies linearly w.r.t. y; Strain ε_{yy} = 0
  - Curvature: $-\mathrm{d}^2 v / \mathrm{d}x^2$
  - Can assume plane stress in z-dir ⇒ basically uniaxial status

$$\sigma_{xx} = E\varepsilon_{xx} = E\varepsilon_0 - Ey\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}$$

- Axial force resultant and bending moment

$$P = \int_A \sigma_{xx}\mathrm{d}A = E\varepsilon_0 \int_A \mathrm{d}A - E\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\cancel{\int_A y\mathrm{d}A}$$

$$M = -\int_A y\sigma_{xx}\mathrm{d}A = -E\varepsilon_0 \cancel{\int_A y\mathrm{d}A} + E\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\boxed{\int_A y^2\mathrm{d}A}$$

Moment of inertia I(x)

$$\Longrightarrow \boxed{\begin{aligned} P &= EA\varepsilon_0 \\ M &= EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \end{aligned}}$$

EA: axial rigidity

EI: flexural rigidity

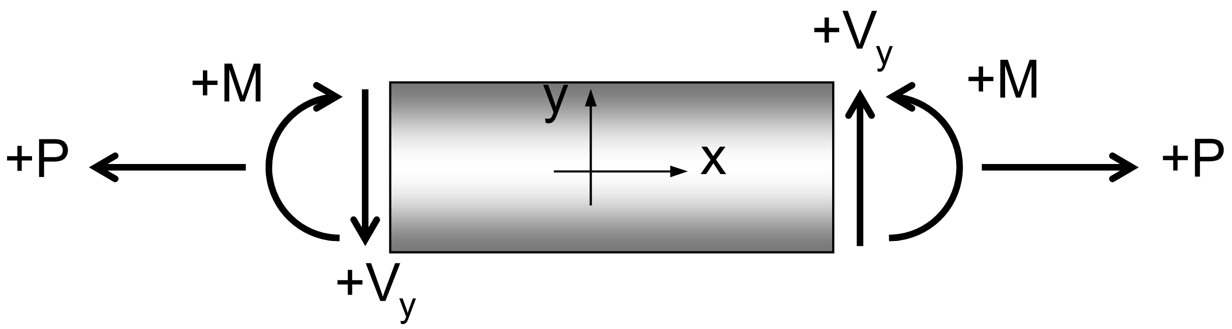
BEAM THEORY *cont.*

- Beam constitutive relation
  - We assume *P* = 0 (We will consider non-zero *P* in the frame element)
  - Moment-curvature relation:

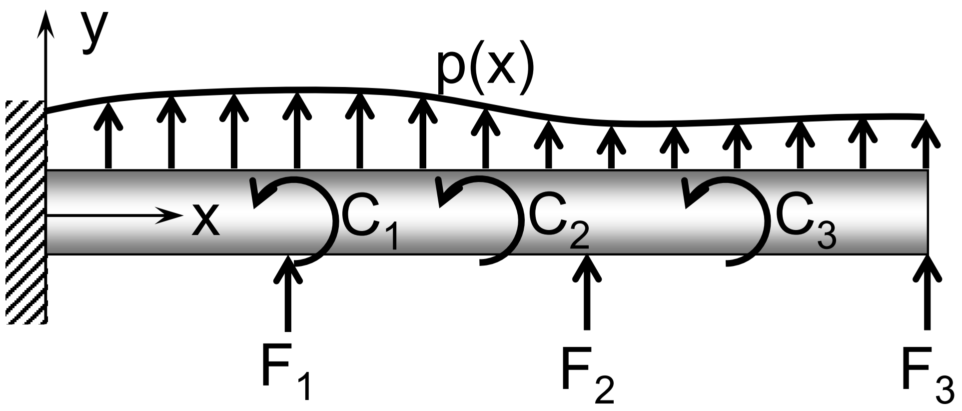
$$\boxed{M = EI\frac{d^2v}{dx^2}}$$

Moment and curvature is linearly dependent

- Sign convention



  - Positive directions for applied loads



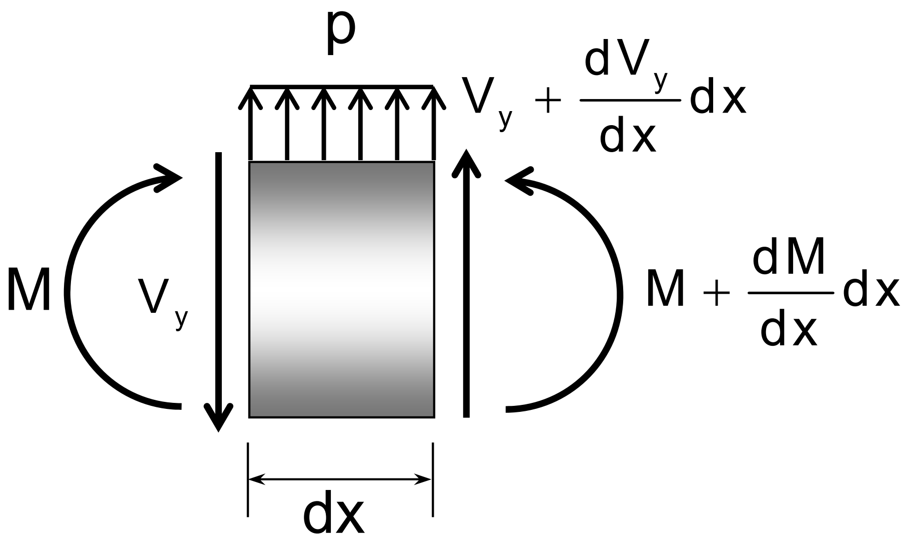
GOVERNING EQUATIONS

- Beam equilibrium equations

$$\sum f_y = 0 \Rightarrow p(x)dx + \left(V_y + \frac{dV_y}{dx}dx \right) - V_y = 0 \quad \Longrightarrow \quad \boxed{\frac{dV_y}{dx} = -p(x)}$$

$$-M + \left(M + \frac{dM}{dx}dx \right) - (pdx)\frac{dx}{2} + V_y dx = 0 \quad \Longrightarrow \quad \boxed{V_y = -\frac{dM}{dx}}$$

  - Combining three equations together: $\boxed{EI\frac{d^4v}{dx^4} = p(x)}$
  - Fourth-order differential equation



STRESS AND STRAIN

- Bending stress

$$\sigma_{xx} = -Ey\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \quad \Longleftrightarrow \quad M = EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}$$

$$\Downarrow$$

$$\boxed{\sigma_{xx}(x,y) = -\frac{M(x)y}{I}}$$

Bending stress

  - This is only non-zero stress component for Euler-Bernoulli beam

- Transverse shear strain

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} = 0 \quad \Longleftarrow \quad u(x,y) = u_0(x) - y\frac{\mathrm{d}v}{\mathrm{d}x}$$

  - Euler beam predicts zero shear strain (approximation)
  - Traditional beam theory says the transverse shear stress is $\tau_{xy} = \frac{VQ}{Ib}$
  - However, this shear stress is in general small compared to the bending stress

POTENTIAL ENERGY

- Potential energy $\Pi = U + V$
- Strain energy
  - Strain energy density

$$U_0 = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} = \frac{1}{2}E(\varepsilon_{xx})^2 = \frac{1}{2}E\left(-y\frac{d^2v}{dx^2}\right)^2 = \frac{1}{2}Ey^2\left(\frac{d^2v}{dx^2}\right)^2$$

  - Strain energy per unit length

$$U_L(x) = \int_A U_0(x,y,z)dA = \int_A \frac{1}{2}Ey^2\left(\frac{d^2v}{dx^2}\right)^2 dA = \frac{1}{2}E\left(\frac{d^2v}{dx^2}\right)^2 \boxed{\int_A y^2 dA}$$

Moment of inertia

$$\boxed{U_L(x) = \frac{1}{2}EI\left(\frac{d^2v}{dx^2}\right)^2}$$

  - Strain energy

$$\boxed{U = \int_0^L U_L(x)dx = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx}$$

POTENTIAL ENERGY *cont.*

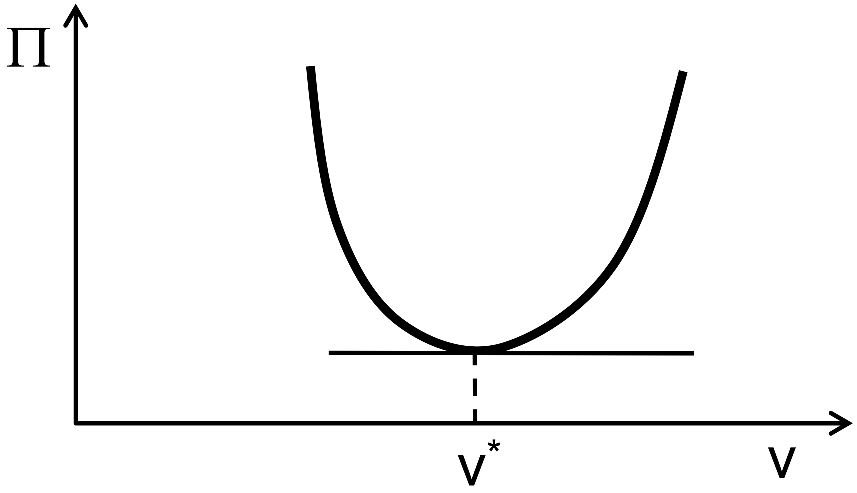
- Potential energy of applied loads

$$V = -\int_0^L p(x)v(x)\,\mathrm{d}x - \sum_{i=1}^{N_F} F_i v(x_i) - \sum_{i=1}^{N_C} C_i \frac{\mathrm{d}v(x_i)}{\mathrm{d}x}$$

- Potential energy

$$\Pi = U + V = \frac{1}{2}\int_0^L EI\left(\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)^2 \mathrm{d}x - \int_0^L p(x)v(x)\,\mathrm{d}x - \sum_{i=1}^{N_F} F_i v(x_i) - \sum_{i=1}^{N_C} C_i \frac{\mathrm{d}v(x_i)}{\mathrm{d}x}$$

  - Potential energy is a function of *v*(*x*) and slope
  - The beam is in equilibrium when Π has its minimum value



$$\frac{\partial \Pi}{\partial v} = 0$$

3.2 RAYLEIGH-RITZ METHOD

RAYLEIGH-RITZ METHOD

1. Assume a deflection shape

$$v(x) = c_1 f_1(x) + c_2 f_2(x) \ldots\ldots + c_n f_n(x)$$

   - Unknown coefficients c_i and known function $f_i(x)$
   - Deflection curve v(x) must satisfy displacement boundary conditions

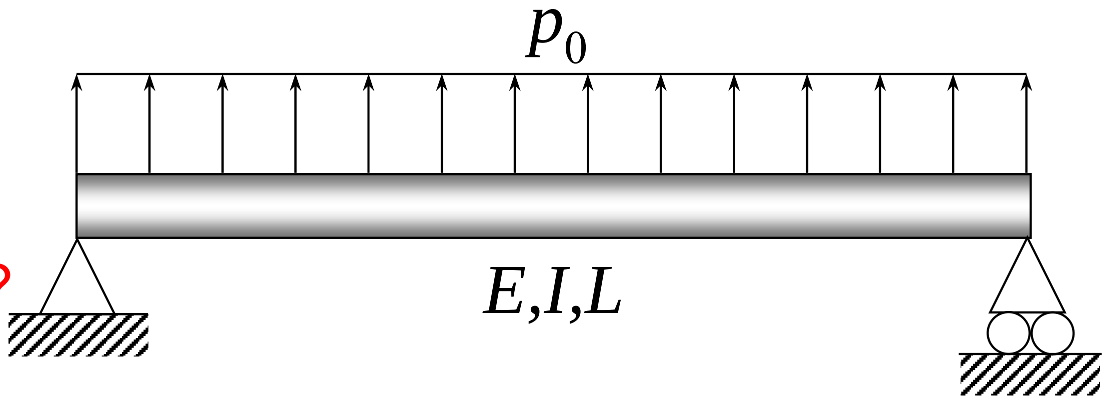
2. Obtain potential energy as function of coefficients

$$\Pi(c_1, c_2, \ldots c_n) = U + V$$

3. Apply the principle of minimum potential energy to determine the coefficients

$$\frac{\partial \Pi}{\partial c_1} = \frac{\partial \Pi}{\partial c_2} = \cdots = \frac{\partial \Pi}{\partial c_n} = 0$$

EXAMPLE – SIMPLY SUPPORTED BEAM



- Assumed deflection curve

$$v(x) = C\sin\frac{\pi x}{L}$$

Satisfying essential BC?

- Strain energy

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{C^2EI\pi^4}{4L^3}$$

- Potential energy of applied loads (no reaction forces)

$$V = -\int_0^L p(x)v(x)dx = -\int_0^L p_0C\sin\frac{\pi x}{L}dx = -\frac{2p_0L}{\pi}C$$

- Potential energy $\Pi = U + V = \frac{EI\pi^4}{4L^3}C^2 - \frac{2p_0L}{\pi}C$

- PMPE: $\frac{d\Pi}{dC} = \frac{EI\pi^4}{2L^3}C - \frac{2p_0L}{\pi} = 0 \quad \Rightarrow \quad C = \frac{4p_0L^4}{EI\pi^5}$

EXAMPLE – SIMPLY SUPPORTED BEAM *cont.*

- Exact vs. approximate deflection at the center

$$C_{\text{approx}} = \frac{p_0 L^4}{76.5EI} \qquad C_{\text{exact}} = \frac{p_0 L^4}{76.8EI}$$

- Approximate bending moment and shear force

$$M(x) = EI\frac{d^2v}{dx^2} = -EIC\frac{\pi^2}{L^2}\sin\frac{\pi x}{L} = -\frac{4p_0L^2}{\pi^3}\sin\frac{\pi x}{L}$$

$$V_y(x) = -EI\frac{d^3v}{dx^3} = EIC\frac{\pi^3}{L^3}\cos\frac{\pi x}{L} = \frac{4p_0L}{\pi^2}\cos\frac{\pi x}{L}$$

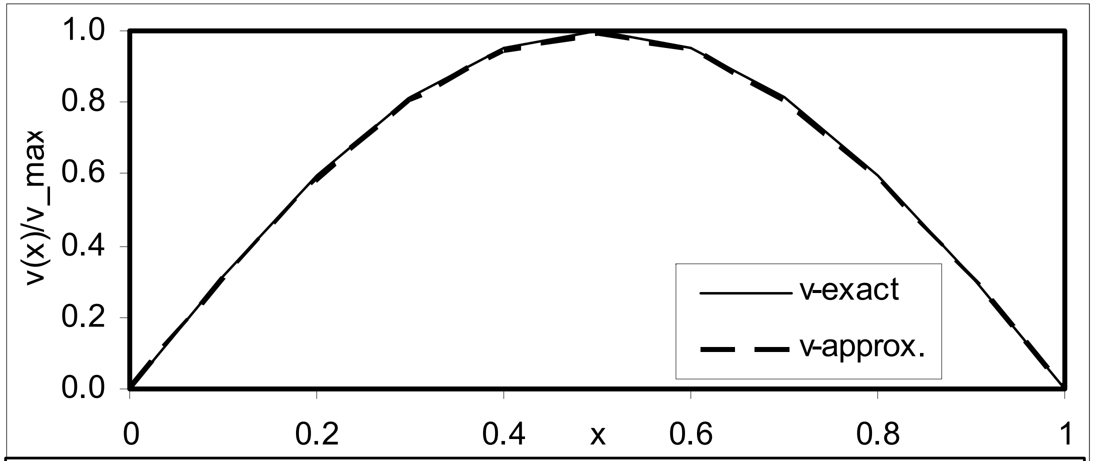
- Exact solutions $v(x) = \frac{1}{EI}\left(\frac{p_0L^3}{24}x - \frac{p_0L}{12}x^3 + \frac{p_0}{24}x^4\right)$

$$M(x) = -\frac{p_0L}{2}x + \frac{p_0}{2}x^2$$

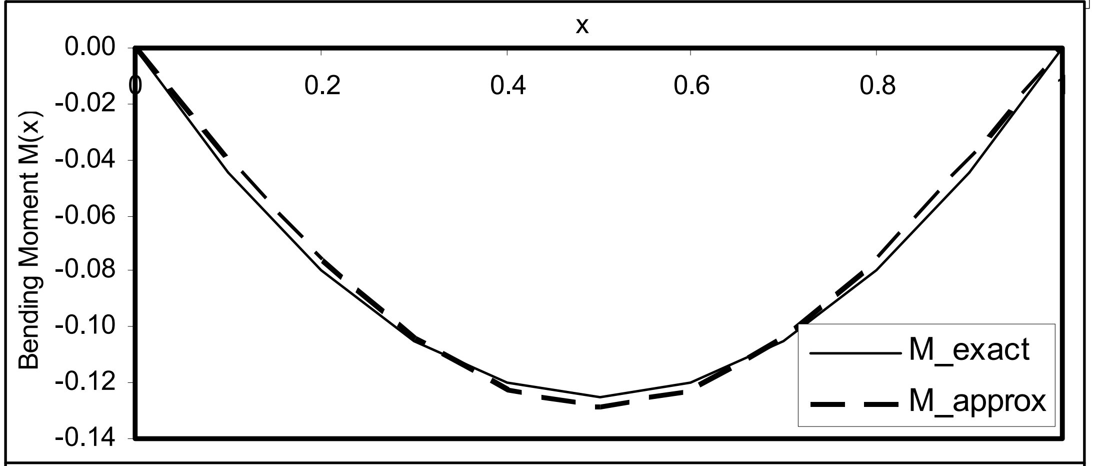
$$V_y(x) = \frac{p_0L}{2} - p_0x$$

EXAMPLE – SIMPLY SUPPORTED BEAM *cont*.

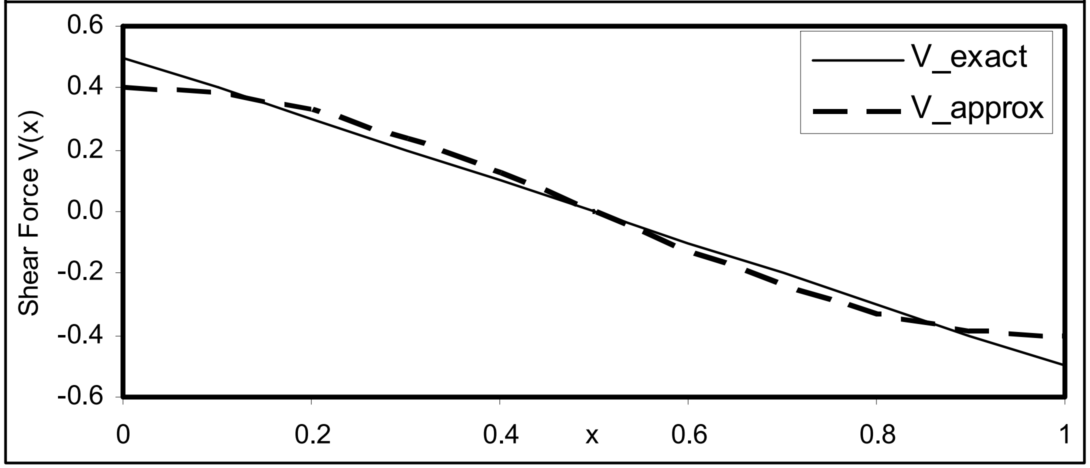
- Deflection



- Bending moment

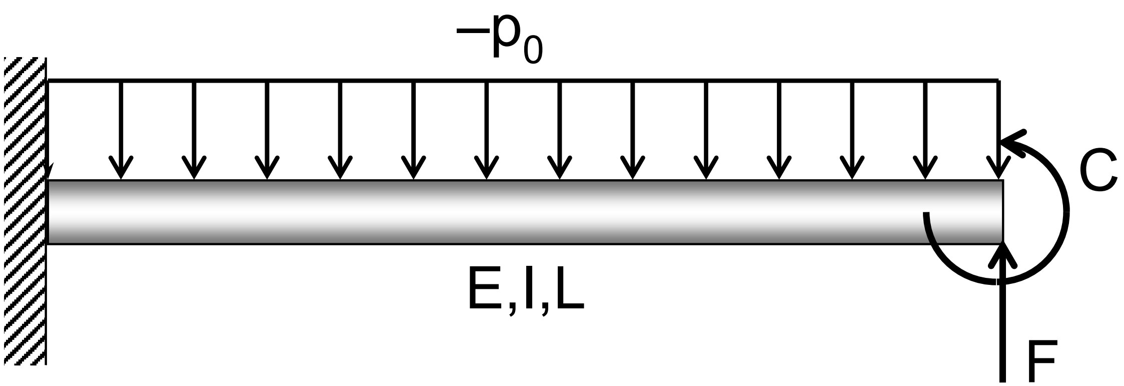


- Shear force



Error increases

EXAMPLE – CANTILEVERED BEAM



- Assumed deflection

$$v(x) = a + bx + c_1x^2 + c_2x^3$$

- Need to satisfy BC

$$v(0) = 0, \mathrm{d}v(0)/\mathrm{d}x = 0 \;\Longrightarrow\; v(x) = c_1x^2 + c_2x^3$$

Need to remove non-admissible trial functions!!

- Strain energy $U = \frac{EI}{2}\int_0^L (2c_1 + 6c_2x)^2\,\mathrm{d}x$

- Potential of loads

$$V(c_1,c_2) = -\int_0^L (-p_0)v(x)\mathrm{d}x - Fv(L) - C\frac{\mathrm{d}v}{\mathrm{d}x}(L)$$

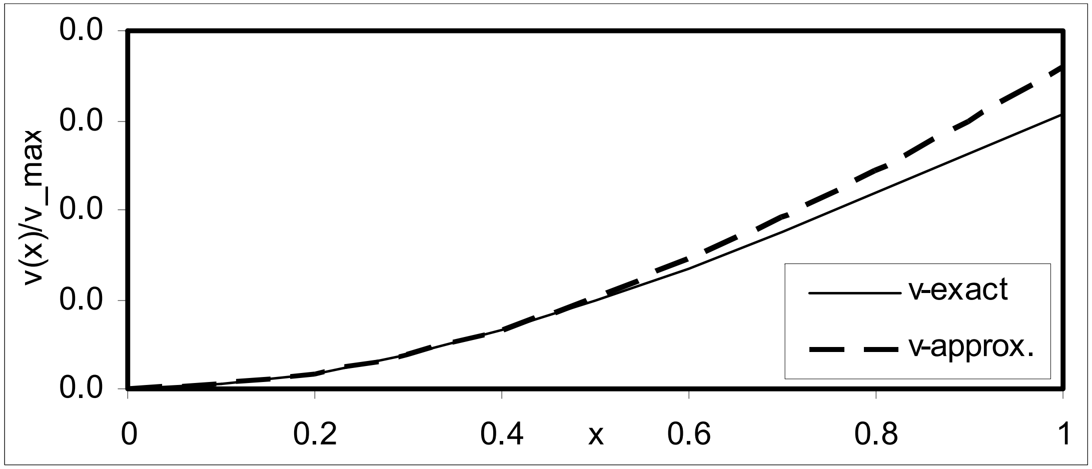
$$= c_1\left(\frac{p_0L^3}{3} - FL^2 - 2CL\right) + c_2\left(\frac{p_0L^4}{4} - FL^3 - 3CL^2\right)$$

EXAMPLE – CANTILEVERED BEAM *cont.*

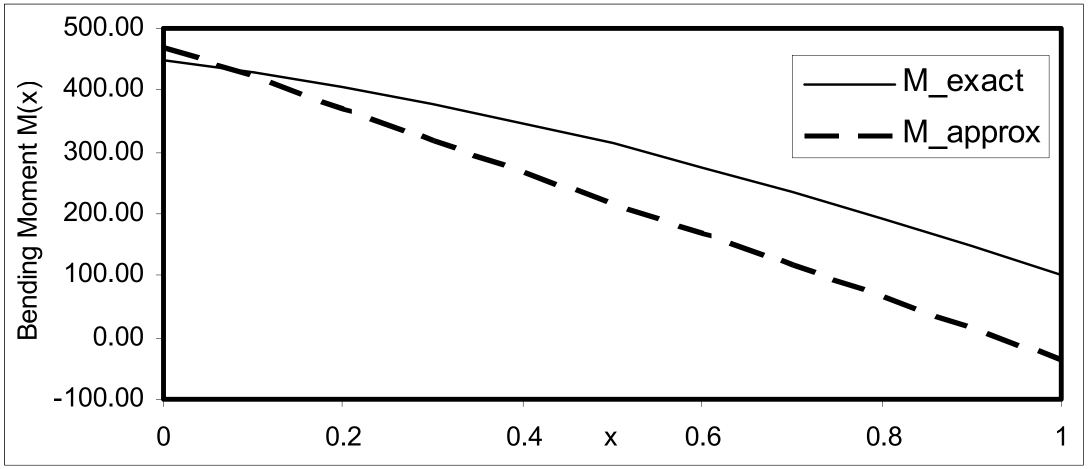
- Derivatives of *U*: $$\frac{\partial U}{\partial c_1} = 2EI\int_0^L (2c_1 + 6c_2 x)\,\mathrm{d}x = EI\left(4Lc_1 + 6L^2 c_2\right)$$
  $$\frac{\partial U}{\partial c_2} = 6EI\int_0^L (2c_1 + 6c_2 x)\,x\mathrm{d}x = EI\left(6L^2 c_1 + 12L^3 c_2\right)$$
- PMPE: $$\frac{\partial \Pi}{\partial c_1} = 0 \qquad EI\left(4Lc_1 + 6L^2 c_2\right) = -\frac{p_0 L^3}{3} + FL^2 + 2CL$$
  $$\Rightarrow$$
  $$\frac{\partial \Pi}{\partial c_2} = 0 \qquad EI\left(6L^2 c_1 + 12L^3 c_2\right) = -\frac{p_0 L^4}{4} + FL^3 + 3CL^2$$
- Solve for c_1 and c_2: $c_1 = 23.75 \times 10^{-3},\ c_2 = -8.417 \times 10^{-3}$
- Deflection curve: $v(x) = 10^{-3}\left(23.75x^2 - 8.417x^3\right)$
- Exact solution: $v(x) = \frac{1}{24EI}\left(5400x^2 - 800x^3 - 300x^4\right)$

EXAMPLE – CANTILEVERED BEAM *cont.*

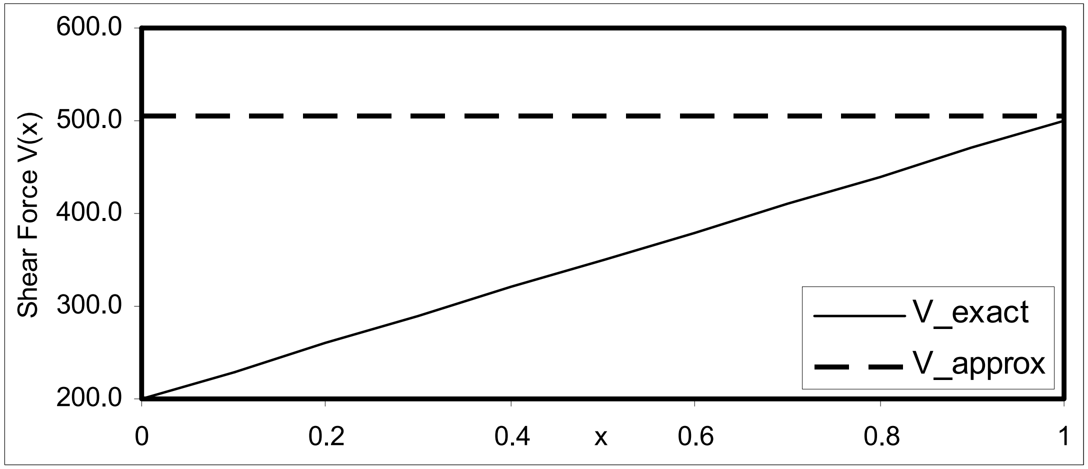
- Deflection

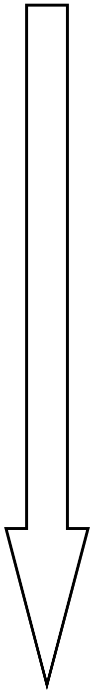


- Bending moment



- Shear force

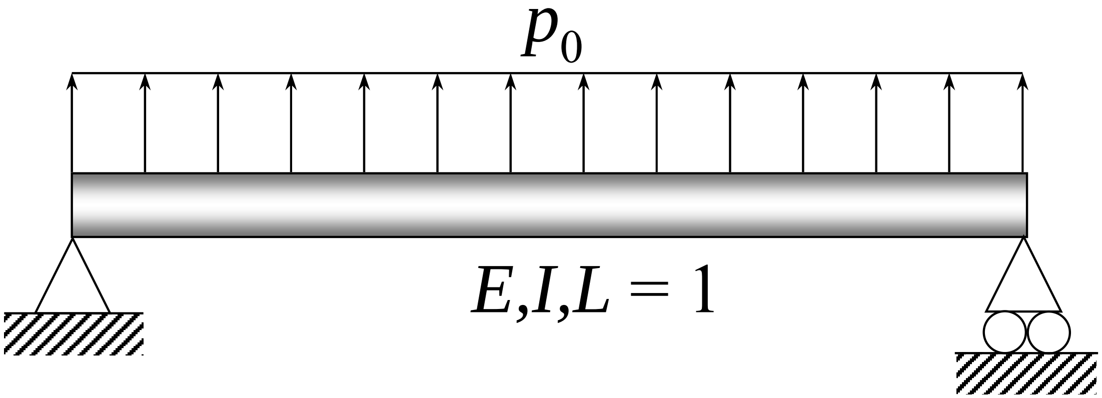




Error increases

Exercise

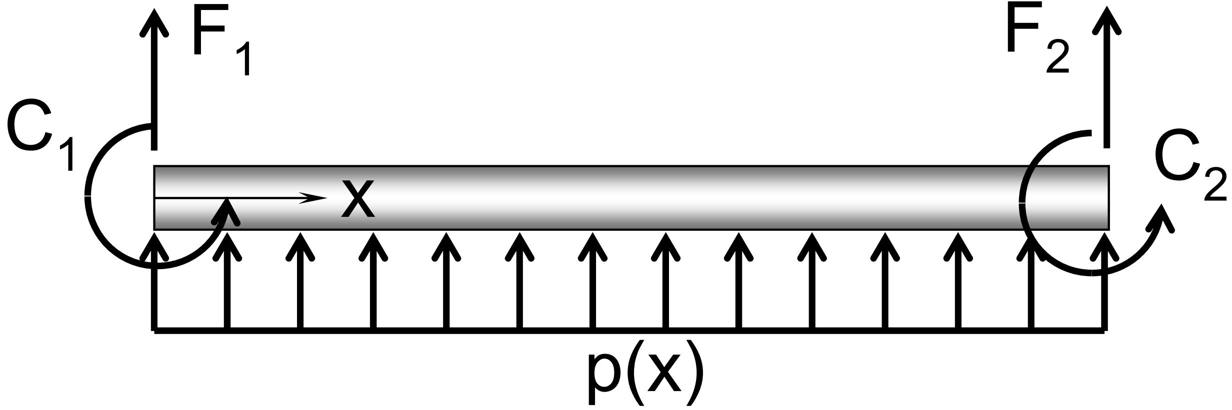
- Calculate the deflection curve v(x), bending moment M(x), and shear force Vy(x) of the simply-supported beam using two basis functions: $f_1(x) = x(1-x),\ f_2(x) = x^2(1-x)$



3.3 FINITE ELEMENT INTERPOLATION

FINITE ELEMENT INTERPOLATION

- Rayleigh-Ritz method approximate solution in the entire beam
  - Difficult to find approx solution that satisfies displacement BC
- Finite element approximates solution in an element
  - Make it easy to satisfy displacement BC using interpolation technique
- Beam element
  - Divide the beam using a set of elements
  - Elements are connected to other elements at nodes
  - Concentrated forces and couples can only be applied at nodes
  - Consider two-node beam element
  - Positive directions for forces and couples
  - Constant or linearly distributed load

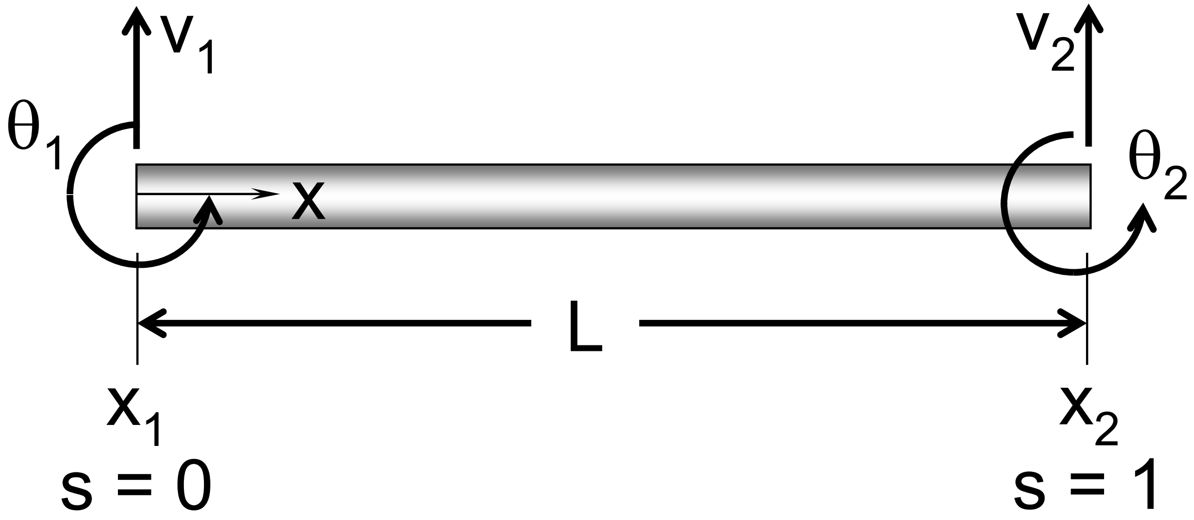


FINITE ELEMENT INTERPOLATION *cont.*

- Nodal DOF of beam element
  - Each node has deflection v and slope θ
  - Positive directions of DOFs
  - Vector of nodal DOFs $\{\mathbf{q}\} = \{v_1 \quad \theta_1 \quad v_2 \quad \theta_2\}^T$
- Scaling parameter s
  - Length L of the beam is scaled to 1 using scaling parameter s

$$s = \frac{x - x_1}{L}, \quad ds = \frac{1}{L}dx,$$

$$dx = L ds, \quad \frac{ds}{dx} = \frac{1}{L}$$



- Will write deflection curve v(s) in terms of s

FINITE ELEMENT INTERPOLATION *cont.*

- Deflection interpolation
  - Interpolate the deflection $v(s)$ in terms of four nodal DOFs
  - Use cubic function: $v(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3$
  - Relation to the slope: $$\theta = \frac{dv}{dx} = \frac{dv}{ds}\frac{ds}{dx} = \frac{1}{L}(a_1 + 2a_2 s + 3a_3 s^2)$$
  - Apply four conditions:

$$v(0) = v_1 \qquad \frac{dv(0)}{dx} = \theta_1 \qquad v(1) = v_2 \qquad \frac{dv(1)}{dx} = \theta_2$$

  - Express four coefficients in terms of nodal DOFs

$$\begin{aligned}
v_1 &= v(0) = a_0 \\
\theta_1 &= \frac{dv}{dx}(0) = \frac{1}{L}a_1 \\
v_2 &= v(1) = a_0 + a_1 + a_2 + a_3 \\
\theta_2 &= \frac{dv}{dx}(1) = \frac{1}{L}(a_1 + 2a_2 + 3a_3)
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
a_0 &= v_1 \\
a_1 &= L\theta_1 \\
a_2 &= -3v_1 - 2L\theta_1 + 3v_2 - L\theta_2 \\
a_3 &= 2v_1 + L\theta_1 - 2v_2 + L\theta_2
\end{aligned}$$

FINITE ELEMENT INTERPOLATION *cont.*

- Deflection interpolation cont.

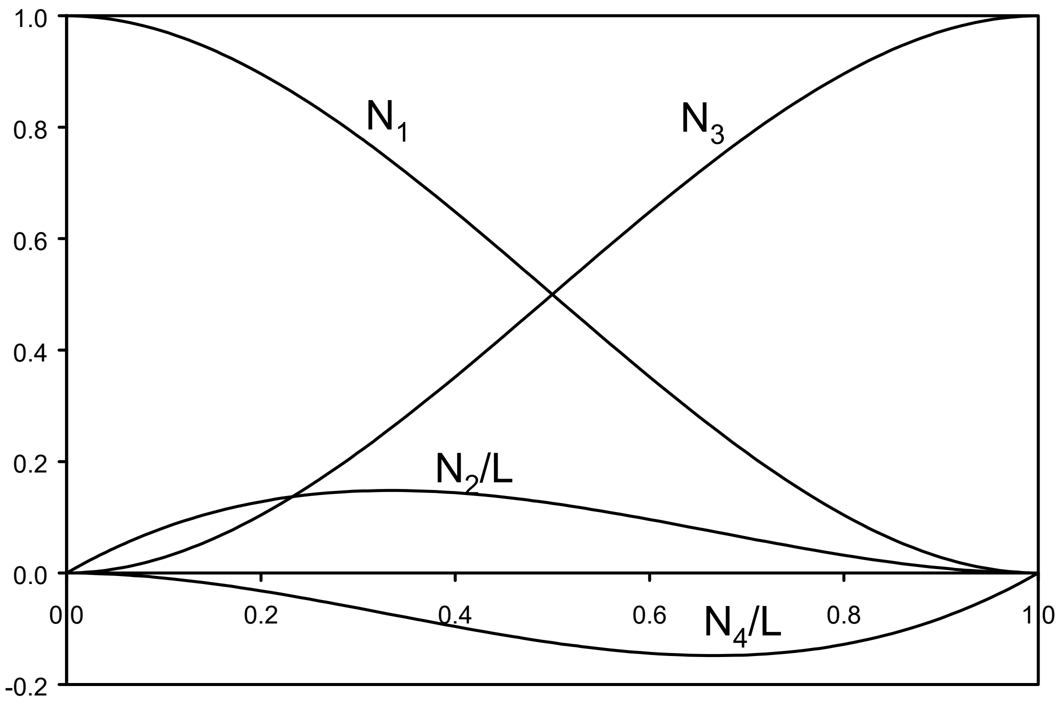
$$v(s) = (1 - 3s^2 + 2s^3)v_1 + L(s - 2s^2 + s^3)\theta_1 + (3s^2 - 2s^3)v_2 + L(-s^2 + s^3)\theta_2$$

$$v(s) = [N_1(s) \quad N_2(s) \quad N_3(s) \quad N_4(s)] \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\boxed{v(s) = \lfloor \mathbf{N} \rfloor \{\mathbf{q}\}}$$

- Shape functions

$$\boxed{\begin{aligned} N_1(s) &= 1 - 3s^2 + 2s^3 \\ N_2(s) &= L(s - 2s^2 + s^3) \\ N_3(s) &= 3s^2 - 2s^3 \\ N_4(s) &= L(-s^2 + s^3) \end{aligned}}$$



  - Hermite polynomials
  - Interpolation property

FINITE ELEMENT INTERPOLATION *cont.*

- Properties of interpolation
  - Deflection is a cubic polynomial (discuss accuracy and limitation)
  - Interpolation is valid within an element, not outside of the element
  - Adjacent elements have continuous deflection and slope
- Approximation of curvature
  - Curvature is second derivative and related to strain and stress

$$\frac{d^2v}{dx^2} = \frac{1}{L^2}\frac{d^2v}{ds^2} = \frac{1}{L^2}[-6+12s, L(-4+6s), 6-12s, L(-2+6s)]\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\boxed{\frac{d^2v}{dx^2} = \frac{1}{L^2}\underset{1\times 4}{\{\mathbf{B}\}^T}\underset{4\times 1}{\{\mathbf{q}\}}}$$

B: strain-displacement vector

  - **B** is linear function of *s* and, thus, the strain and stress
  - Alternative expression: $\frac{d^2v}{dx^2} = \frac{1}{L^2}\underset{1\times 4}{\{\mathbf{q}\}^T}\underset{4\times 1}{\{\mathbf{B}\}}$
  - If the given problem is linearly varying curvature, the approximation is accurate; if higher-order variation of curvature, then it is approximate

FINITE ELEMENT INTERPOLATION *cont*.

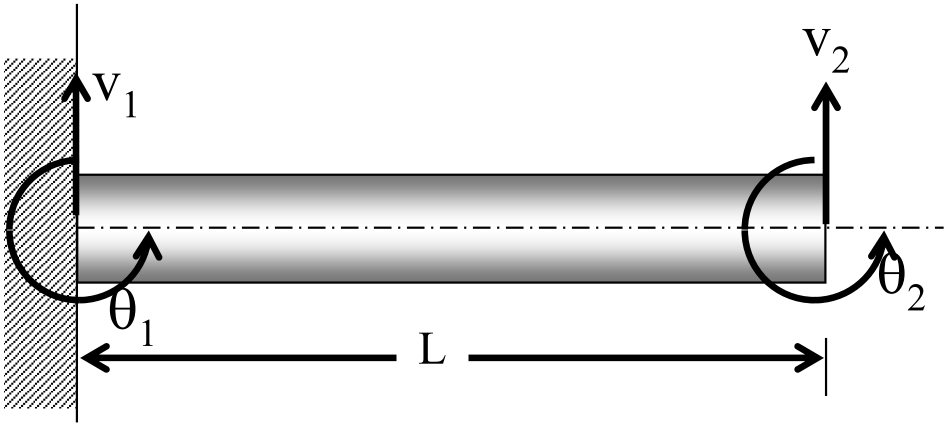
- Approximation of bending moment and shear force

$$M(s) = EI\frac{d^2v}{dx^2} = \frac{EI}{L^2}\{\mathbf{B}\}^T\{\mathbf{q}\}$$ Linear

$$V_y = -\frac{dM}{dx} = -EI\frac{d^3v}{dx^3} = \frac{EI}{L^3}[-12 \quad -6L \quad 12 \quad -6L]\{\mathbf{q}\}$$ Constant

  - Stress is proportional to M(s); M(s) is linear; stress is linear, too
  - Maximum stress always occurs at the node
  - Bending moment and shear force are not continuous between adjacent elements

EXAMPLE – INTERPOLATION



- Cantilevered beam
- Given nodal DOFs

$$\{\mathbf{q}) = \{0, 0, -0.1, -0.2\}^T$$

- Deflection and slope at x = 0.5L
- Parameter s = 0.5 at x = 0.5L
- Shape functions: $N_1(\tfrac{1}{2}) = \dfrac{1}{2},\quad N_2(\tfrac{1}{2}) = \dfrac{L}{8},\quad N_3(\tfrac{1}{2}) = \dfrac{1}{2},\quad N_4(\tfrac{1}{2}) = -\dfrac{L}{8}$
- Deflection at s = 0.5:

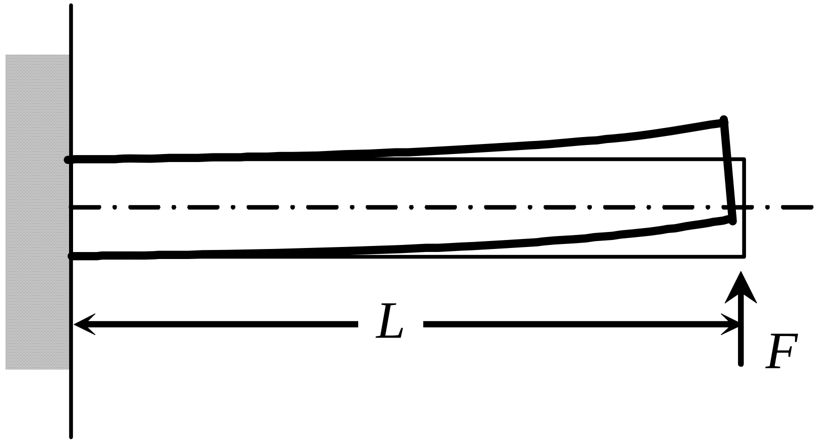
$$\begin{aligned} v(\tfrac{1}{2}) &= N_1(\tfrac{1}{2})v_1 + N_2(\tfrac{1}{2})\theta_1 + N_3(\tfrac{1}{2})v_2 + N_4(\tfrac{1}{2})\theta_2 \\ &= \frac{1}{2}\times 0 + \frac{L}{8}\times 0 + \frac{1}{2}\times v_2 - \frac{L}{8}\times\theta_2 = \frac{v_2}{2} - \frac{L\theta_2}{8} = -0.025 \end{aligned}$$

- Slope at s = 0.5:

$$\frac{dv}{dx} = \frac{1}{L}\frac{dv}{ds} = \frac{1}{L}\left(v_1\frac{dN_1}{ds} + \theta_1\frac{dN_2}{ds} + v_2\frac{dN_3}{ds} + \theta_2\frac{dN_4}{ds}\right)$$

$$= v_1\frac{1}{L}(-6s + 6s^2) + \theta_1\left(1 - 4s + 3s^2\right) + v_2\frac{1}{L}(6s - 6s^2) + \theta_2\left(-2s + 3s^2\right) = -0.1$$

EXAMPLE



- A beam finite element with length L

$$v_1 = 0, \theta_1 = 0, v_2 = \frac{L^3}{3EI}, \theta_2 = \frac{L^2}{2EI}$$

- Calculate v(s)

$$v(s) = N_1(s)\cancel{v_1} + N_2(s)\cancel{\theta_1} + N_3(s)v_2 + N_4(s)\theta_2$$

$$v(s) = (3s^2 - 2s^3)v_2 + L(-s^2 + s^3)\theta_2$$

- Bending moment

$$\begin{aligned} M(s) &= EI\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = \frac{EI}{L^2}\frac{\mathrm{d}^2v}{\mathrm{d}s^2} = \frac{EI}{L^2}\left[(6-12s)v_2 + L(-2+6s)\theta_2\right] \\ &= \frac{EI}{L^2}\left[(6-12s)\frac{L^3}{3EI} + L(-2+6s)\frac{L^2}{2EI}\right] \\ &= L(1-s) = (L-x) \end{aligned}$$

Bending moment cause by unit force at the tip

Exercise

- Calculate the beam shape functions when the natural coordinate is given as s = [-1, +1]
  - Hint: Assume $v(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3$ determine 4 coefficients using the following conditions:

$$v(-1) = v_1 \qquad \frac{\mathrm{d}v(-1)}{\mathrm{d}x} = \theta_1 \qquad v(1) = v_2 \qquad \frac{\mathrm{d}v(1)}{\mathrm{d}x} = \theta_2$$

3.4 FE EQUATION FOR BEAM ELEMENT

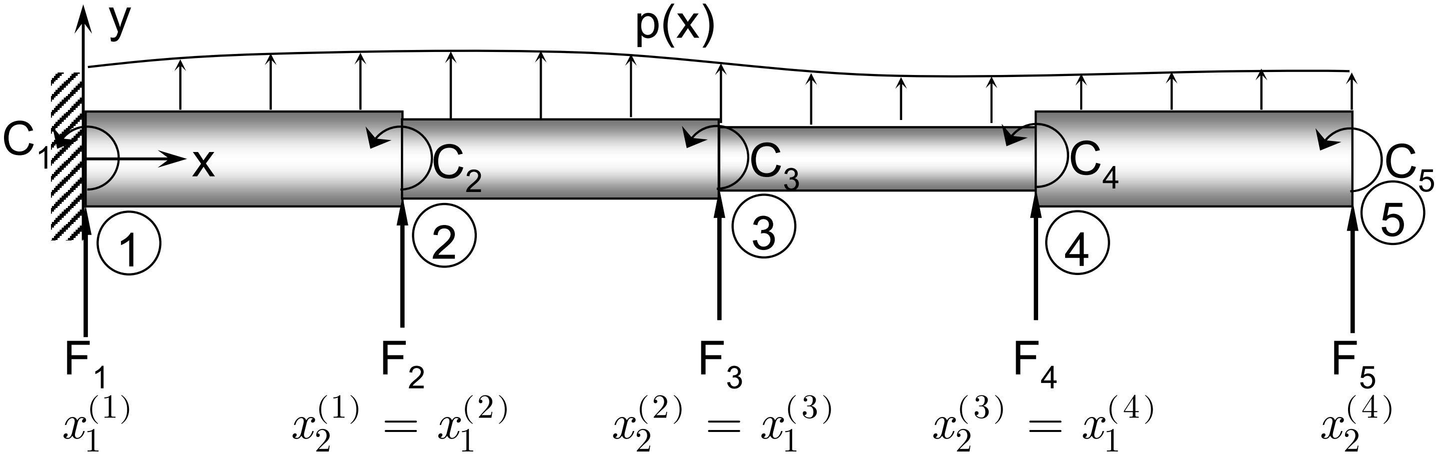
FINITE ELEMENT EQUATION FOR BEAM

- Finite element equation using PMPE
  - A beam is divided by NEL elements with constant sections
- Strain energy
  - Sum of each element's strain energy

$$U = \int_0^{L_T} U_L(x)dx = \sum_{e=1}^{NEL} \int_{x_1^{(e)}}^{x_2^{(e)}} U_L(x)dx = \sum_{e=1}^{NEL} U^{(e)}$$

  - Strain energy of element (e)

$$U^{(e)} = EI \int_{x_1^{(e)}}^{x_2^{(e)}} \frac{1}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{EI}{L^3}\int_0^1 \frac{1}{2}\left(\frac{d^2v}{ds^2}\right)^2 ds$$



FE EQUATION FOR BEAM *cont.*

- Strain energy *cont.*
  - Approximate curvature in terms of nodal DOFs

$$\left(\frac{d^2v}{ds^2}\right)^2 = \left(\frac{d^2v}{ds^2}\right)\left(\frac{d^2v}{ds^2}\right) = \underset{1\times 4}{\{\mathbf{q}^{(e)}\}^T}\underset{4\times 1}{\{\mathbf{B}\}}\underset{1\times 4}{\{\mathbf{B}\}^T}\underset{4\times 1}{\{\mathbf{q}^{(e)}\}}$$

  - Approximate element strain energy in terms of nodal DOFs

$$U^{(e)} = \frac{1}{2}\{\mathbf{q}^{(e)}\}^T\left[\frac{EI}{L^3}\int_0^1\{\mathbf{B}\}\{\mathbf{B}\}^T\,ds\right]^{(e)}\{\mathbf{q}^{(e)}\} = \frac{1}{2}\{\mathbf{q}^{(e)}\}^T[\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\}$$

- Stiffness matrix of a beam element

$$[\mathbf{k}^{(e)}] = \frac{EI}{L^3}\int_0^1\begin{bmatrix}-6+12s\\ L(-4+6s)\\ 6-12s\\ L(-2+6s)\end{bmatrix}\begin{bmatrix}-6+12s & L(-4+6s) & 6-12s & L(-2+6s)\end{bmatrix}ds$$

FE EQUATION FOR BEAM *cont.*

- Stiffness matrix of a beam element

$$[\mathbf{k}^{(e)}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Symmetric, positive semi-definite

Proportional to EI

Inversely proportional to L

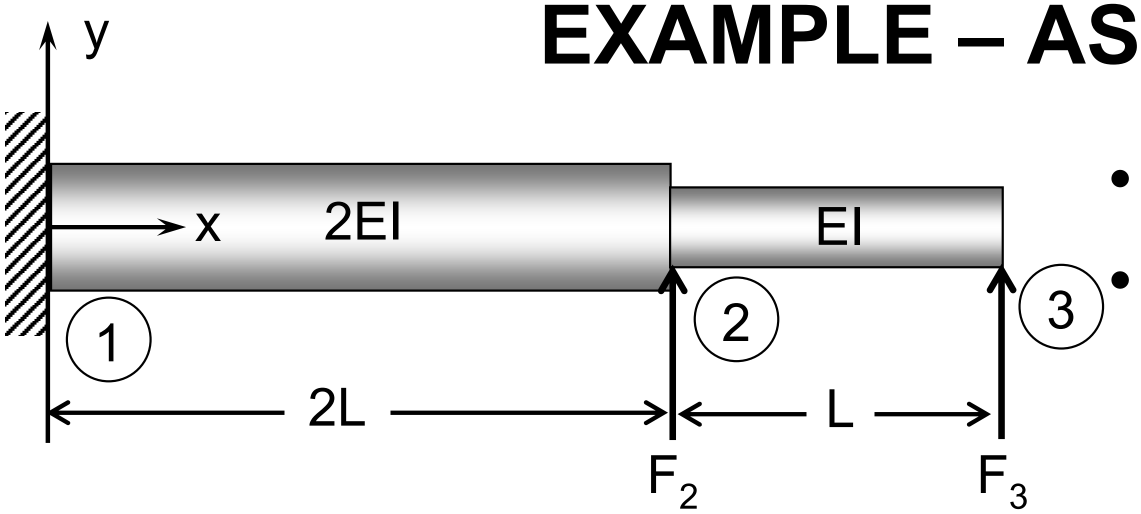
- Strain energy *cont.*

$$U = \sum_{e=1}^{NEL} U^{(e)} = \frac{1}{2}\sum_{e=1}^{NEL} \{\mathbf{q}^{(e)}\}^T [\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\}$$

  - Assembly

$$U = \frac{1}{2}\{\mathbf{Q}_s\}^T [\mathbf{K}_s]\{\mathbf{Q}_s\}$$

EXAMPLE – ASSEMBLY



- Two elements
- Global DOFs

$$\{\mathbf{Q}_s\}^T = \{v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \quad v_3 \quad \theta_3\}$$

$$[\mathbf{k}^{(1)}] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccc} 3 & 3L & -3 & 3L \\ 3L & 4L^2 & -3L & 2L^2 \\ -3 & -3L & 3 & -3L \\ 3L & 2L^2 & -3L & 4L^2 \end{array}\right] \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

$$[\mathbf{k}^{(2)}] = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \left[\begin{array}{cccc} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right] \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

$$[\mathbf{K}_s] = \frac{EI}{L^3}\begin{bmatrix} 3 & 3L & -3 & 3L & 0 & 0 \\ 3L & 4L^2 & -3L & 2L^2 & 0 & 0 \\ -3 & -3L & 15 & 3L & -12 & 6L \\ 3L & 2L^2 & 3L & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

FE EQUATION FOR BEAM *cont.*

- Potential energy of applied loads
  - Concentrated forces and couples

$$V = -\sum_{i=1}^{ND}\left(F_i v_i + C_i \theta_i\right) \quad \Longrightarrow \quad V = -\lfloor v_1\ \theta_1\ v_2 \ldots\ldots \theta_{ND} \rfloor \begin{Bmatrix} F_1 \\ C_1 \\ F_2 \\ \vdots \\ C_{ND} \end{Bmatrix} = \boxed{-\{\mathbf{Q}_s\}^T\{\mathbf{F}_s\}}$$

  - Distributed load (Work-equivalent nodal forces)

$$V = -\sum_{e=1}^{NEL}\int_{x_1^{(e)}}^{x_2^{(e)}} p(x)v(x)dx = \sum_{e=1}^{NEL} V^{(e)} \quad \Longrightarrow \quad V^{(e)} = \int_{x_1^{(e)}}^{x_2^{(e)}} p(x)v(x)dx = L^{(e)}\int_0^1 p(s)v(s)ds$$

$$\begin{aligned} V^{(e)} &= L^{(e)}\int_0^1 p(s)\left(v_1N_1 + \theta_1N_2 + v_2N_3 + \theta_2N_4\right)ds \\ &= v_1\left(L^{(e)}\int_0^1 p(s)N_1ds\right) + \theta_1\left(L^{(e)}\int_0^1 p(s)N_2ds\right) + v_2\left(L^{(e)}\int_0^1 p(s)N_3ds\right) + \theta_2\left(L^{(e)}\int_0^1 p(s)N_4ds\right) \\ &= v_1F_1^{(e)} + \theta_1C_1^{(e)} + v_2F_2^{(e)} + \theta_2C_2^{(e)} \end{aligned}$$

EXAMPLE – WORK-EQUIVALENT NODAL FORCES

- Uniformly distributed load

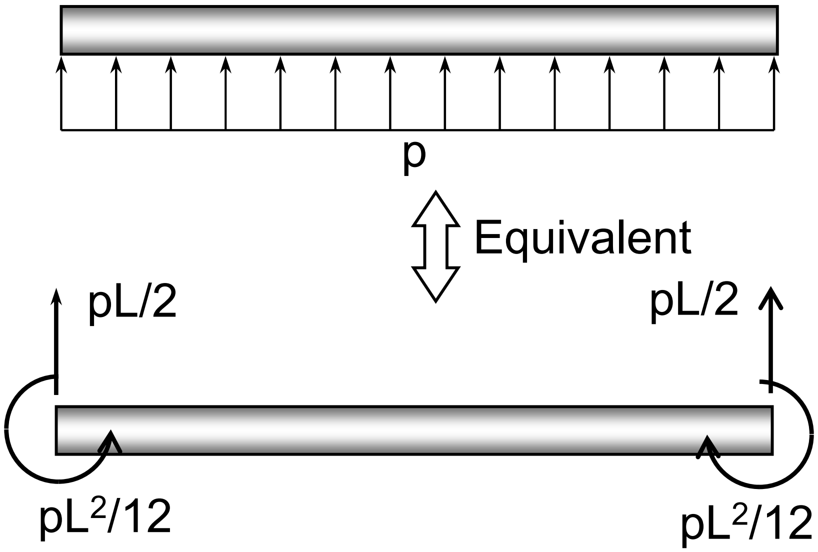
$$F_1 = pL\int_0^1 N_1(s)ds = pL\int_0^1 (1-3s^2+2s^3)ds = \frac{pL}{2}$$

$$C_1 = pL\int_0^1 N_2(s)ds = pL^2\int_0^1 (s-2s^2+s^3)ds = \frac{pL^2}{12}$$

$$F_2 = pL\int_0^1 N_3(s)ds = pL\int_0^1 (3s^2-2s^3)ds = \frac{pL}{2}$$

$$C_2 = pL\int_0^1 N_4(s)ds = pL^2\int_0^1 (-s^2+s^3)ds = -\frac{pL^2}{12}$$

$$\{\mathbf{F}\}^T = \left\{ \frac{pL}{2} \quad \frac{pL^2}{12} \quad \frac{pL}{2} \quad -\frac{pL^2}{12} \right\}$$



FE EQUATION FOR BEAM *cont.*

- Finite element equation for beam

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} pL/2 \\ pL^2/12 \\ pL/2 \\ -pL^2/12 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ C_1 \\ F_2 \\ C_2 \end{Bmatrix}$$

  - One beam element has four variables
  - When there is no distributed load, $p = 0$
  - Applying boundary conditions is identical to truss element
  - At each DOF, either displacement (v or θ) or force (F or C) must be known, not both
  - Use standard procedure for assembly, BC, and solution

PRINCIPLE OF MINIMUM POTENTIAL ENERGY

- Potential energy (quadratic form)

$$\Pi = U + V = \frac{1}{2}\{\mathbf{Q}_s\}^T[\mathbf{K}_s]\{\mathbf{Q}_s\} - \{\mathbf{Q}_s\}^T\{\mathbf{F}_s\}$$

- PMPE
  - Potential energy has its minimum when

$$\boxed{[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\}}$$

[**K**$_s$] is symmetric & PSD

- Applying BC
  - The same procedure with truss elements (striking-the-rows and striking-he-columns)

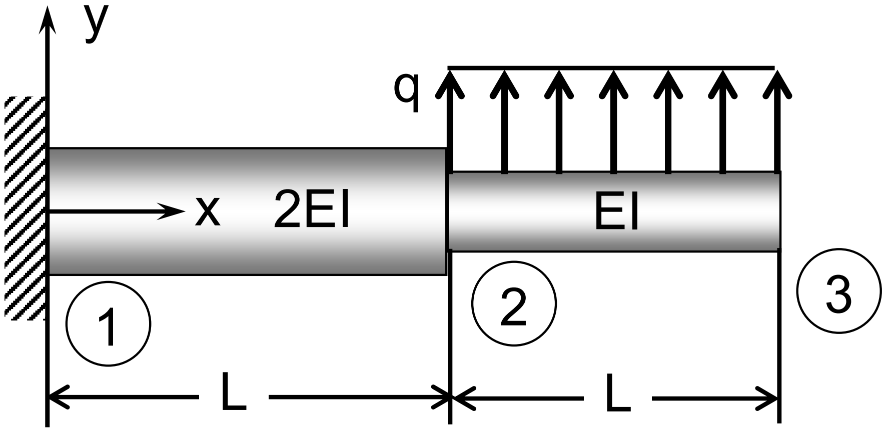
$$[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\}$$

[**K**] is symmetric & PD

- Solve for unknown nodal DOFs {**Q**}

Exercise

- Calculate the tip deflection of cantilevered beam shown in the figure. Use L=1m, EI = 10^4Nm^2, and q=100N/m.



3.5 BENDING MOMENT AND SHEAR FORCE

BENDING MOMENT & SHEAR FORCE

- Bending moment

$$M(s) = EI\frac{d^2v}{dx^2} = \frac{EI}{L^2}\frac{d^2v}{ds^2} = \frac{EI}{L^2}\{\mathbf{B}\}^T\{\mathbf{q}\}$$

  - Linearly varying along the beam span
- Shear force

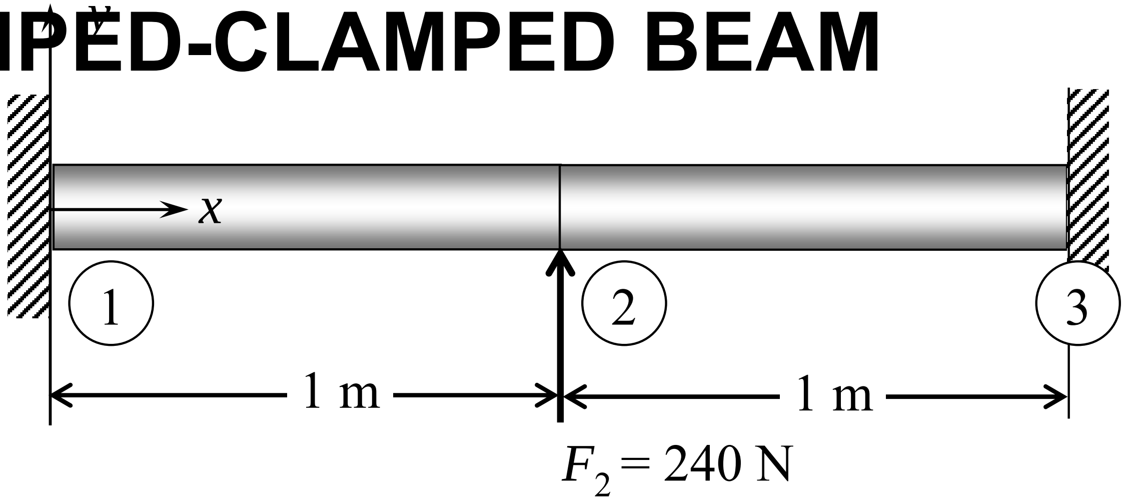
$$V_y(s) = -\frac{dM}{dx} = -EI\frac{d^3v}{dx^3} = -\frac{EI}{L^3}\frac{d^3v}{ds^3} = \frac{EI}{L^3}\begin{bmatrix} -12 & -6L & 12 & -6L \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

  - Constant
  - When true moment is not linear and true shear is not constant, many elements should be used to approximate it
- Bending stress $\sigma_x = -\frac{My}{I}$
- Shear stress for rectangular section

$$\tau_{xy}(y) = \frac{1.5V_y}{bh}\left(1 - \frac{4y^2}{h^2}\right)$$

EXAMPLE – CLAMPED-CLAMPED BEAM

- Determine deflection & slope at x = 0.5, 1.0, 1.5 m
- Element stiffness matrices



$$[\mathbf{k}^{(1)}] = 1000\begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccc} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{array}\right] \end{array}\begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

$$[\mathbf{k}^{(2)}] = 1000\begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \left[\begin{array}{cccc} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{array}\right] \end{array}\begin{array}{c} \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

$$1000\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ C_1 \\ 240 \\ 0 \\ F_3 \\ C_3 \end{Bmatrix}$$

EXAMPLE – CLAMPED-CLAMPED BEAM *cont.*

- Applying BC

$$1000\begin{bmatrix}24 & 0\\ 0 & 8\end{bmatrix}\begin{Bmatrix}v_2\\ \theta_2\end{Bmatrix}=\begin{Bmatrix}240\\ 0\end{Bmatrix} \quad\Longrightarrow\quad \begin{matrix} v_2 = 0.01 \\ \theta_2 = 0.0\end{matrix}$$

- At x = 0.5 $\Longrightarrow$ s = 0.5 and use element 1

$$v(\tfrac{1}{2}) = v_1N_1(\tfrac{1}{2}) + \theta_1N_2(\tfrac{1}{2}) + v_2N_3(\tfrac{1}{2}) + \theta_2N_4(\tfrac{1}{2}) = 0.01\times N_3(\tfrac{1}{2}) = 0.005\,\mathrm{m}$$

$$\theta(\tfrac{1}{2}) = \frac{1}{L^{(1)}}v_2\left.\frac{dN_3}{ds}\right|_{s=\frac{1}{2}} = 0.015\,\mathrm{rad}$$

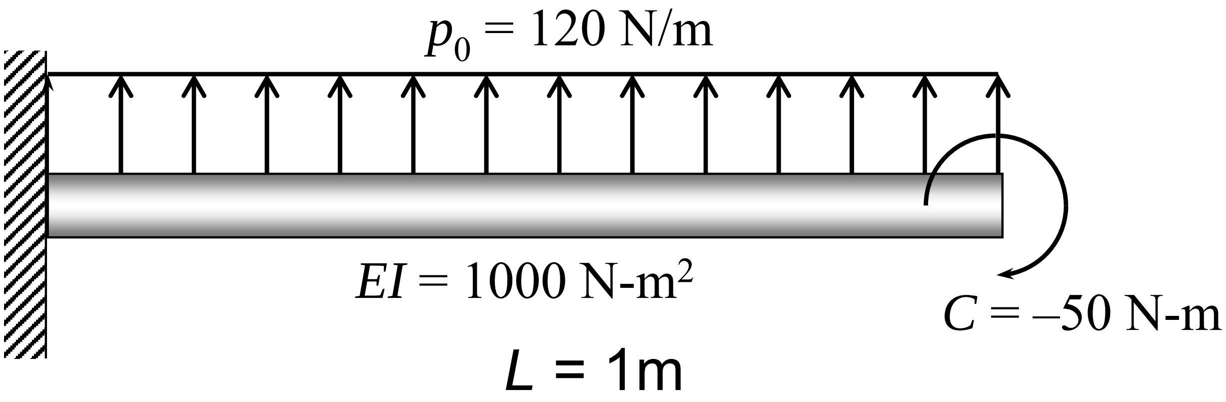
- At x = 1.0 $\Longrightarrow$ either s = 1 (element 1) or s = 0 (element 2)

$$v(1) = v_2N_3(1) = 0.01\times N_3(1) = 0.01\mathrm{m} \qquad v(0) = v_2N_1(0) = 0.01\times N_1(0) = 0.01\mathrm{m}$$

$$\theta(1) = \frac{1}{L^{(1)}}v_2\left.\frac{dN_3}{ds}\right|_{s=1} = 0.0\,\mathrm{rad} \qquad \theta(0) = \frac{1}{L^{(2)}}v_2\left.\frac{dN_1}{ds}\right|_{s=0} = 0.0\,\mathrm{rad}$$

Will this solution be accurate or approximate?

EXAMPLE – CANTILEVERED BEAM



- One beam element
- No assembly required
- Element stiffness

$$[\mathbf{K}_s] = 1000 \left[\begin{array}{cccc} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{array}\right] \begin{array}{l} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

- Work-equivalent nodal forces

$$\begin{Bmatrix} F_{1e} \\ C_{1e} \\ F_{2e} \\ C_{2e} \end{Bmatrix} = p_0 L \int_0^1 \begin{Bmatrix} 1-3s^2+2s^3 \\ (s-2s^2+s^3)L \\ 3s^2-2s^3 \\ (-s^2+s^3)L \end{Bmatrix} ds = p_0 L \begin{Bmatrix} 1/2 \\ L/12 \\ 1/2 \\ -L/12 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 10 \\ 60 \\ -10 \end{Bmatrix}$$

EXAMPLE – CANTILEVERED BEAM *cont.*

- FE matrix equation

$$1000\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 + 60 \\ C_1 + 10 \\ 60 \\ -10 - 50 \end{Bmatrix}$$

- Applying BC

$$1000\begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 60 \\ -60 \end{Bmatrix} \quad \Rightarrow \quad \begin{aligned} v_2 &= -0.01\,\mathrm{m} \\ \theta_2 &= -0.03\,\mathrm{rad} \end{aligned}$$

- Deflection curve: $v(s) = -0.01N_3(s) - 0.03N_4(s) = -0.01s^3$
- Exact solution: $v(x) = 0.005(x^4 - 4x^3 + x^2)$

EXAMPLE – CANTILEVERED BEAM *cont*.

- Support reaction (From assembled matrix equation)

$$1000\left(-12v_2+6\theta_2\right)=F_1+60 \qquad\qquad F_1=-120\,\text{N}$$

⇒

$$1000\left(-6v_2+2\theta_2\right)=C_1+10 \qquad\qquad C_1=-10\,\text{N}\cdot\text{m}$$

- Bending moment

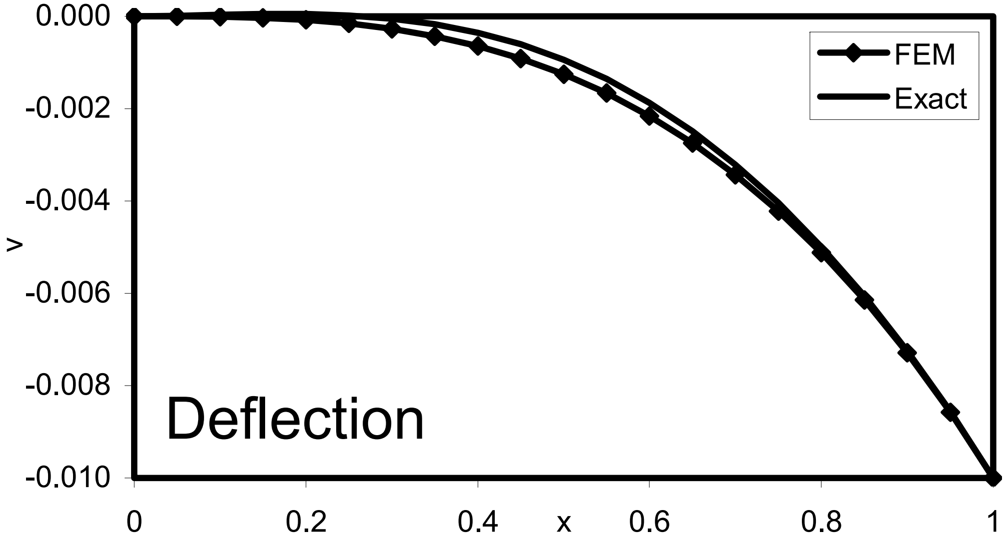
$$\begin{aligned}M(s)&=\frac{EI}{L^2}\lfloor \mathbf{B} \rfloor\{\mathbf{q}\}\\&=\frac{EI}{L^2}\left[(-6+12s)v_1+L(-4+6s)\theta_1+(6-12s)v_2+L(-2+6s)\theta_2\right]\\&=1000[-0.01(6-12s)-0.03(-2+6s)]\\&=-60s\,\text{N}\cdot\text{m}\end{aligned}$$

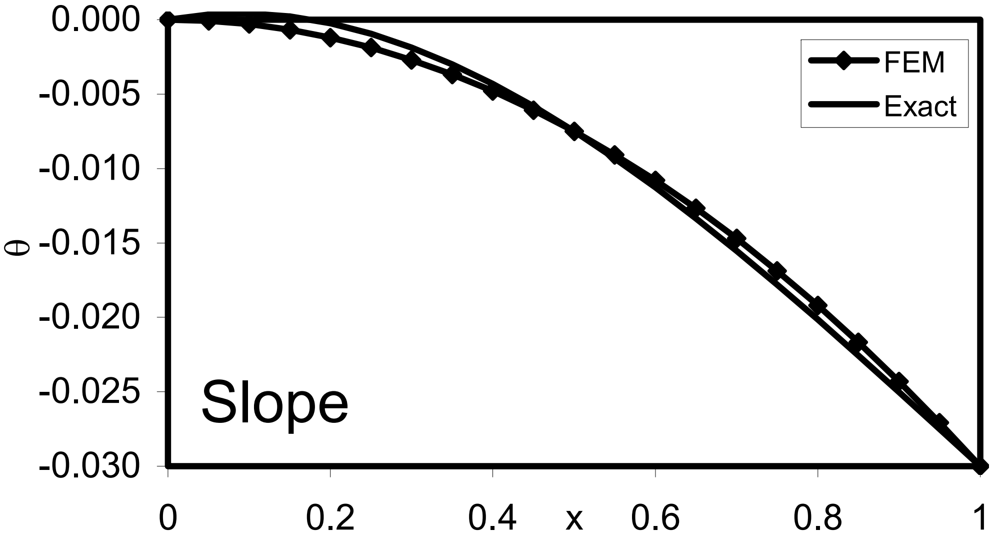
- Shear force

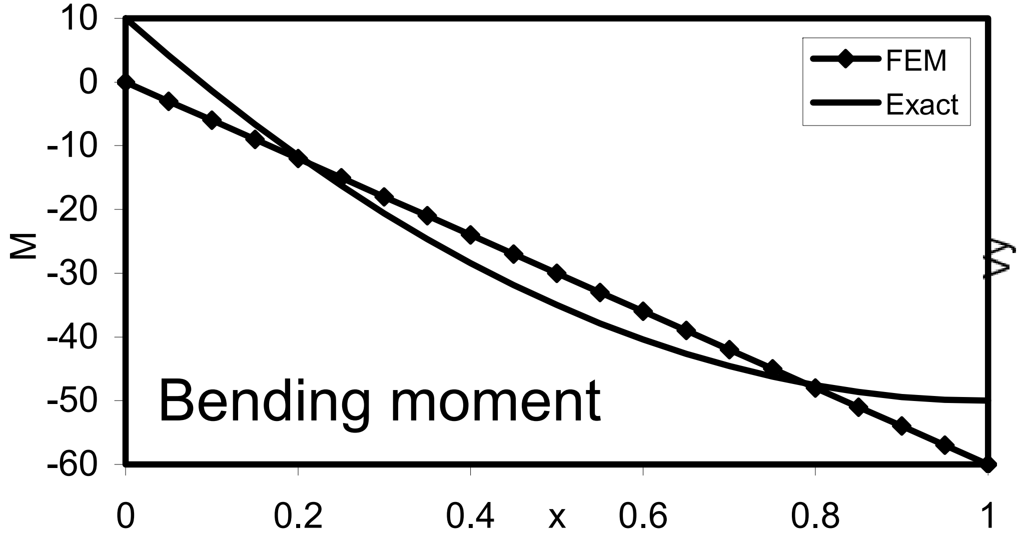
$$\begin{aligned}V_y&=-\frac{EI}{L^3}\left[12v_1+6L\theta_1-12v_2+6L\theta_2\right]\\&=-1000[-12\times(-0.01)+6(-0.03)]\\&=60\,\text{N}\end{aligned}$$

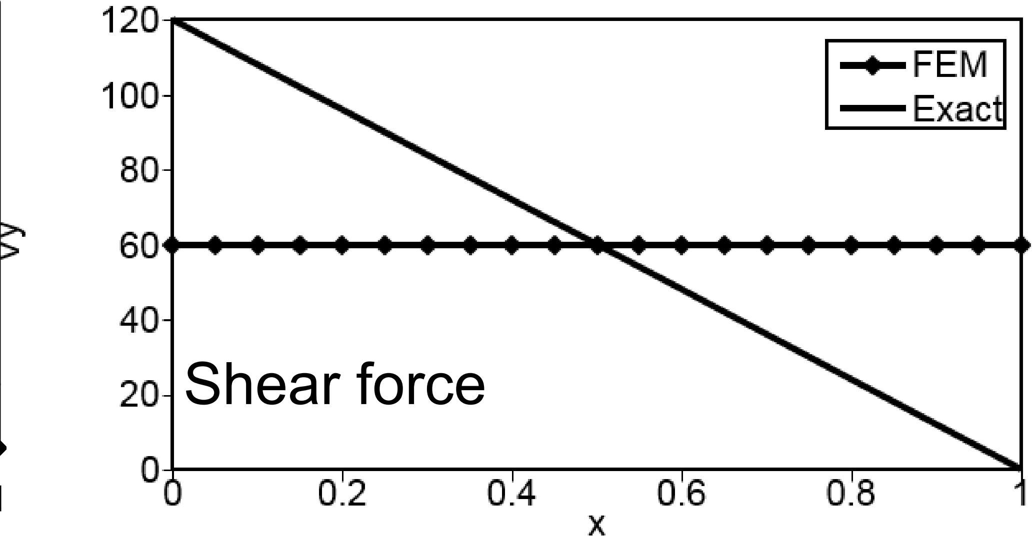
EXAMPLE – CANTILEVERED BEAM *cont.*

- Comparisons







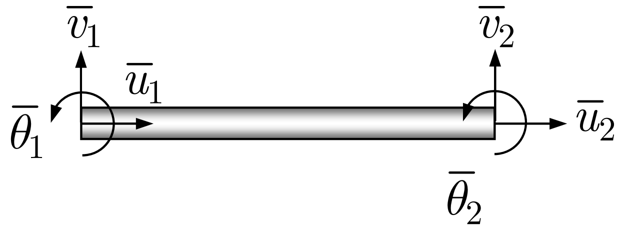


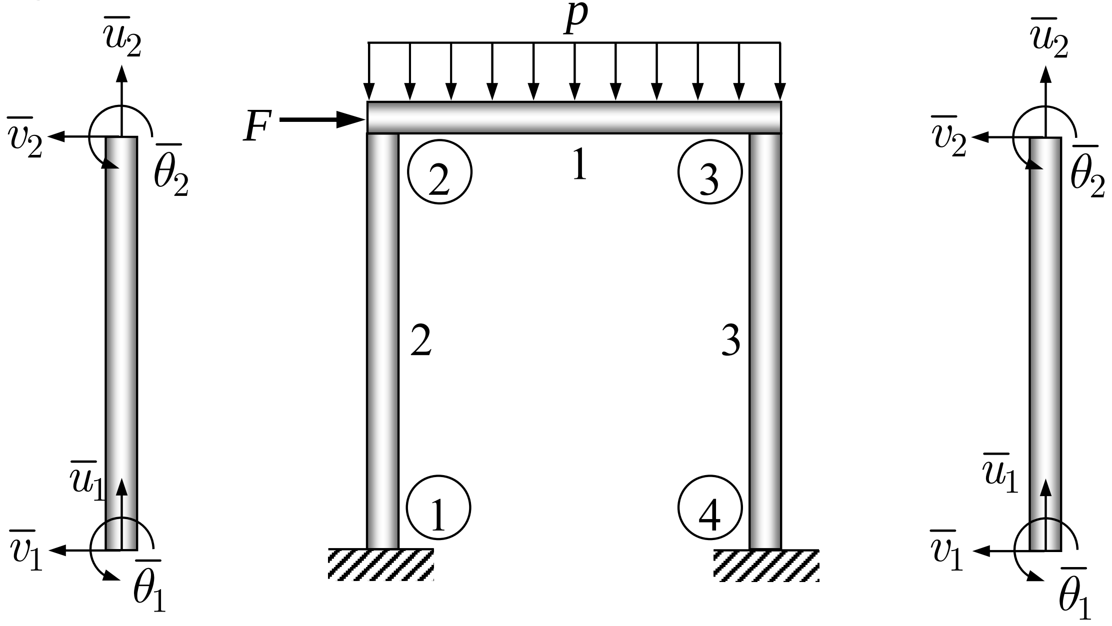
3.6 PLANE FRAME

PLANE FRAME ELEMENT

- Beam
  - Vertical deflection and slope. No axial deformation
- Frame structure
  - Can carry axial force, transverse shear force, and bending moment (Beam + Truss)
- Assumption
  - Axial and bending effects are uncoupled
  - Reasonable when deformation is small
- 3 DOFs per node

  $\{u_i, v_i, \theta_i\}$
- Need coordinate transformation like plane truss





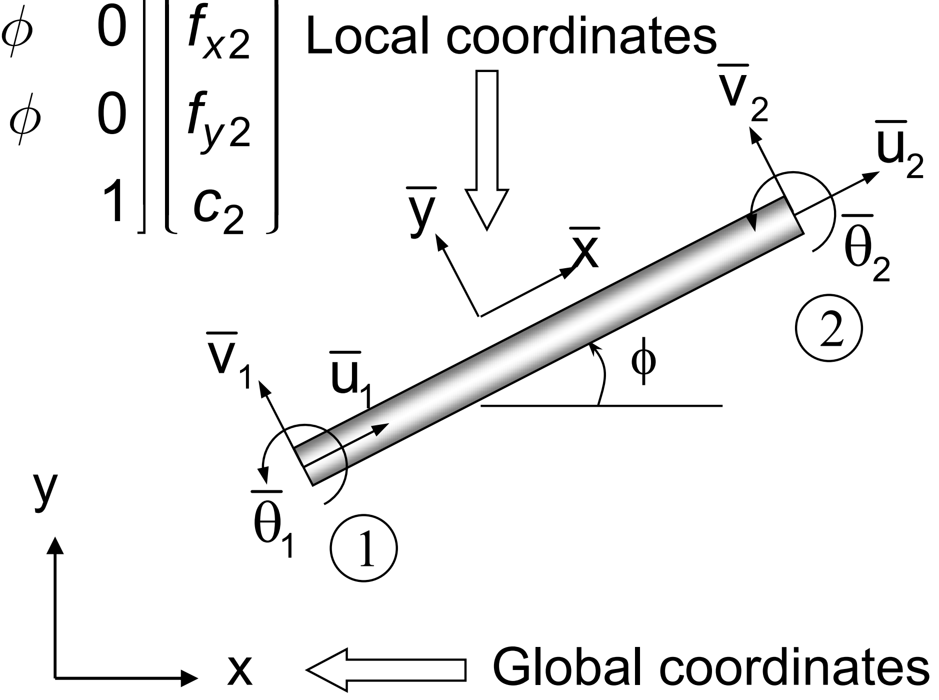
PLANE FRAME ELEMENT *cont.*

- Element-fixed local coordinates $\bar{x}-\bar{y}$
- Local DOFs $\{\bar{u},\bar{v},\bar{\theta}\}$ Local forces $\{f_{\bar{x}}, f_{\bar{y}}, \bar{c}\}$
- Transformation between local and global coord.

$$\begin{Bmatrix} f_{\bar{x}1} \\ f_{\bar{y}1} \\ \bar{c}_1 \\ f_{\bar{x}2} \\ f_{\bar{y}2} \\ \bar{c}_2 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{y1} \\ c_1 \\ f_{x2} \\ f_{y2} \\ c_2 \end{Bmatrix}$$

$$\boxed{\{\bar{\mathbf{f}}\} = [\mathbf{T}]\{\mathbf{f}\}}$$

$$\boxed{\{\bar{\mathbf{q}}\} = [\mathbf{T}]\{\mathbf{q}\}}$$



PLANE FRAME ELEMENT *cont.*

- Axial deformation (in local coord.)

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{Bmatrix} f_{\bar{x}1} \\ f_{\bar{x}2} \end{Bmatrix}$$

- Beam bending

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} f_{\bar{y}1} \\ \bar{c}_1 \\ f_{\bar{y}2} \\ \bar{c}_2 \end{Bmatrix}$$

- Basically, it is equivalent to overlapping a beam with a bar
- A frame element has 6 DOFs

PLANE FRAME ELEMENT *cont.*

- Element matrix equation (local coord.)

$$\begin{bmatrix} a_1 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & 12a_2 & 6La_2 & 0 & -12a_2 & 6La_2 \\ 0 & 6La_2 & 4L^2a_2 & 0 & -6La_2 & 2L^2a_2 \\ -a_1 & 0 & 0 & a_1 & 0 & 0 \\ 0 & -12a_2 & -6La_2 & 0 & 12a_2 & -6La_2 \\ 0 & 6La_2 & 2L^2a_2 & 0 & -6La_2 & 4L^2a_2 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} \bar{f}_{x1} \\ \bar{f}_{y1} \\ \bar{c}_1 \\ \bar{f}_{x2} \\ \bar{f}_{y2} \\ \bar{c}_2 \end{Bmatrix}$$

$$a_1 = \frac{EA}{L}$$

$$a_2 = \frac{EI}{L^3}$$

$$\boxed{[\bar{\mathbf{k}}]\{\bar{\mathbf{q}}\} = \{\bar{\mathbf{f}}\}}$$

- Element matrix equation (global coord.)

$$[\bar{\mathbf{k}}][\mathbf{T}]\{\mathbf{q}\} = [\mathbf{T}]\{\mathbf{f}\} \quad \Longrightarrow \quad [\mathbf{T}]^{\mathrm{T}}[\bar{\mathbf{k}}][\mathbf{T}]\{\mathbf{q}\} = \{\mathbf{f}\} \quad \Longrightarrow \quad [\mathbf{k}]\{\mathbf{q}\} = \{\mathbf{f}\}$$

$$\boxed{[\mathbf{k}] = [\mathbf{T}]^{\mathrm{T}}[\bar{\mathbf{k}}][\mathbf{T}]}$$

- Same procedure for assembly and applying BC

PLANE FRAME ELEMENT *cont.*

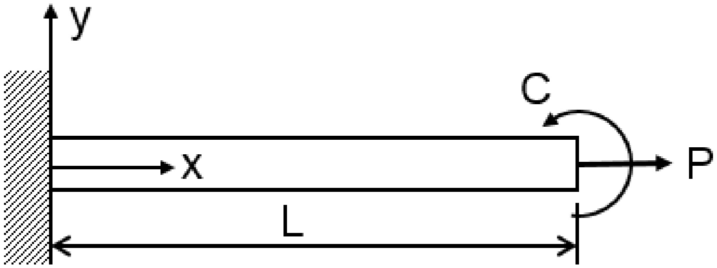
- Calculation of element forces
  - **Element forces can only be calculated in the local coordinate**
  - Extract element DOFs $\{\mathbf{q}\}$ from the global DOFs $\{\mathbf{Q}_s\}$
  - Transform the element DOFs to the local coordinate $\{\bar{\mathbf{q}}\} = [\mathbf{T}]\{\mathbf{q}\}$
  - Then, use 1D bar and beam formulas for element forces
  - Axial force $P = \dfrac{AE}{L}(\bar{u}_2 - \bar{u}_1)$
  - Bending moment $M(s) = \dfrac{EI}{L^2}\{\mathbf{B}\}^{\mathrm{T}}\{\bar{\mathbf{q}}\}$
  - Shear force $V_y(s) = \dfrac{EI}{L^3}[-12 \quad -6L \quad 12 \quad -6L]\{\bar{\mathbf{q}}\}$
- Other method:

$$\begin{Bmatrix} -V_{\bar{y}1} \\ -\bar{M}_1 \\ +V_{\bar{y}2} \\ \bar{M}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix}$$

3.8 BUCKLING OF BEAMS

Review of Buckling of a Beam

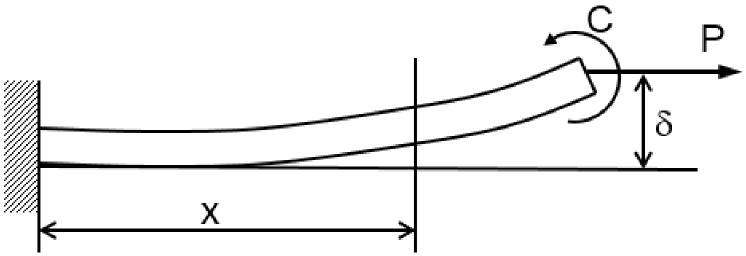
- Tip deflection of a cantilevered beam
  - Without P, $v(L) = \dfrac{CL^2}{2EI}$

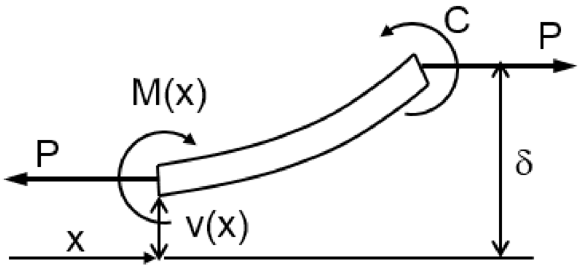


  - Axial tension makes it difficult to bend
  - Axial compression makes the deflection larger
- Free-body diagram with P
  - Bending moment at x: $M(x) = C - \boxed{P(\delta - v(x))} \longrightarrow$ Normally ignored in beam bending

$$M(x) = EI\frac{d^2v}{dx^2} = C - P(\delta - v)$$

$$\Rightarrow \boxed{EI\frac{d^2v}{dx^2} - Pv = C - P\delta}$$ (2nd-order D.E.)





Beam Deflection under Tensile Force

- Solution of $EI\dfrac{d^2v}{dx^2} - Pv = C - P\delta$

$$v(x) = A\sinh\lambda x + B\cosh\lambda x - \left(\frac{C}{P} - \delta\right) \qquad \lambda^2 = \frac{P}{EI}$$

- BCs to determine A & B
  - v(0) = 0 & dv(0)/dx = 0

$$v(x) = \left(\frac{C}{P} - \delta\right)(\cosh\lambda x - 1)$$

- Tip deflection δ = v(L)

$$\delta = v(L) = \frac{C}{P}\left(1 - \frac{1}{\cosh\lambda L}\right), \quad P > 0$$

- As $P\rightarrow\infty$, $\lambda\rightarrow\infty$ and $\delta\rightarrow0$. When $P\rightarrow0$, $\lambda\rightarrow0$, and $\delta \rightarrow \dfrac{CL^2}{2EI}$

Beam Deflection under Compressive Force

- When P < 0 (compressive force)

$$v(x) = \left(\frac{C}{|P|} + \delta\right)(1 - \cos\lambda x), \quad \lambda = \sqrt{\frac{|P|}{EI}}, \quad P < 0$$

- Tip deflection

$$\delta = v(L) = \frac{C}{|P|}\left(\frac{1}{\cos\lambda L} - 1\right), \quad P < 0$$

  - Tip deflection is unbounded when $\lambda L \to \pi / 2$

$$\lambda = \sqrt{\frac{|P|}{EI}} = \frac{\pi}{2L} \quad \text{or} \quad P = P_c = \frac{\pi^2 EI}{4L^2} \approx 2.47\frac{EI}{L^2}$$

- P_c : critical load for buckling of the beam
  - P_c is independent of C
  - Depend on EI and L → structural property

P > 0: stress stiffening

P < 0: stress softening

Energy Method for Beam Buckling

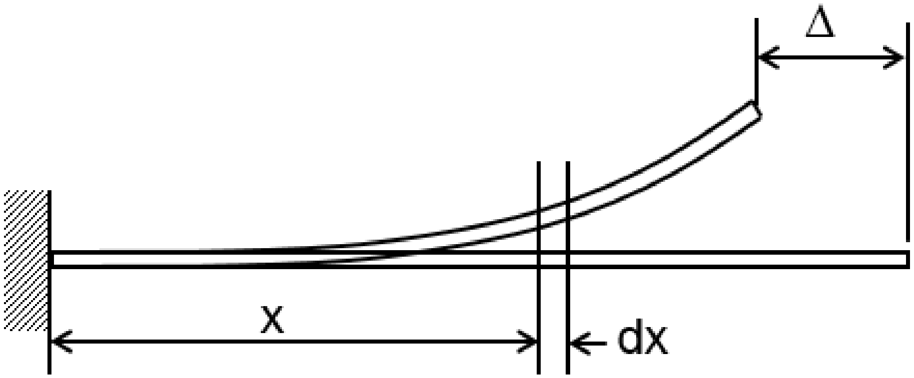
- Shortening of beam due to coupling of P and the flexural deformation
  - Assume no stretching due to P
  - Axial deformation Δ due to bending (positive for shortening)

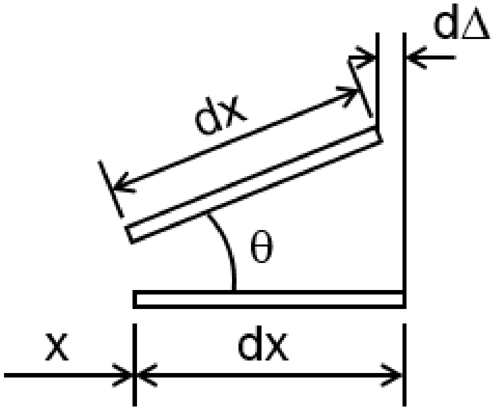
$$d\Delta = dx(1-\cos\theta) \approx 2dx\sin^2\frac{\theta}{2}$$

$$\sin\theta \approx \frac{dv}{dx}$$

$$d\Delta \approx \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx \quad \Longrightarrow \quad \Delta = \int_0^L d\Delta = \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$$

End shortening of the beam





Rayleigh-Ritz Method for Buckling

- Approximate the deflection of the beam $v(x) = Ax^2$
  - Satisfies essential BC
- Strain energy

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx = 2EILA^2$$

- Potential energy of applied loads

$$V = -C\frac{dv}{dx}\bigg|_{x=L} - (P)(-\Delta)$$

$-\Delta$: Same dir. with positive P

$$V = -2CLA + P\frac{1}{2}\int_0^L (2Ax)^2\, dx = -2CLA + \frac{2}{3}PL^3A^2$$

- Total potential energy

$$\Pi = U + V = 2EILA^2 - 2CLA + \frac{2}{3}PL^3A^2$$

Rayleigh-Ritz Method for Buckling

- PMPE:

$$\frac{d\Pi}{dA} = 0 \Rightarrow \left(4EILA - 2CL + \frac{4}{3}PL^3A\right) = 0 \Rightarrow A = \frac{C}{2EI + 2PL^2 / 3}$$

- Tip deflection (tensile force)

$$\delta = v(L) = AL^2 = \frac{CL^2}{2EI + 2PL^2 / 3}, \quad P > 0$$

- Tip deflection (compressive force)

$$\delta = \frac{CL^2}{2EI - 2|P|L^2 / 3}, \quad P < 0$$

- Critical load (unbounded deflection)

$$P_{cr} = 3\frac{EI}{L^2}$$

  - About 20% larger than exact value $P_{cr}^{\text{exact}} \approx 2.47\frac{EI}{L^2}$

Rayleigh-Ritz Method for Buckling

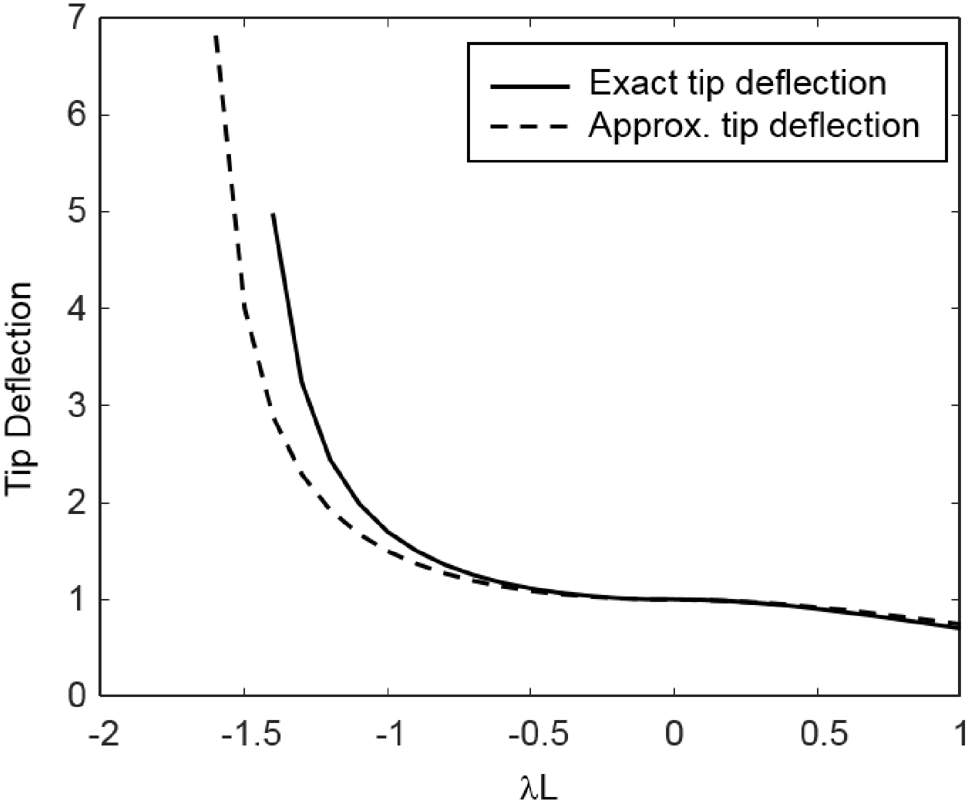
- Non-dimensionalization

$$\bar{\delta} = \frac{\delta}{CL^2/2EI}$$

$$= \frac{1}{1+0.333(\lambda L)^2}, \quad \lambda L > 0$$

$$= \frac{1}{1-0.333(\lambda L)^2}, \quad \lambda L < 0$$

Unbounded when $\lambda L \to \sqrt{3}$



- Non-dimensionalized exact solution

$$\bar{\delta}_{exact} = \frac{2}{(\lambda L)^2}\left(1 - \frac{1}{\cosh \lambda L}\right), \quad \lambda L > 0$$

Unbounded when $\lambda L \to \pi/2$

$$= \frac{2}{(\lambda L)^2}\left(\frac{1}{\cos \lambda L} - 1\right), \quad \lambda L < 0$$

FE Method for Buckling

- Already have matrix form of [**K**]{**Q**} = {**F**}. So only new term…
- Potential (work done) by axial load

$$V_{inc} = -\sum_{e=1}^{NE} P^{(e)}(-\Delta^{(e)})$$

$P^{(e)}$: element axial force

Need to express $\Delta^{(e)}$ in terms of nodal DOFs

- Axial shortening

$$\Delta^{(e)} = \frac{1}{2}\int_{x_i}^{x_j}\left(\frac{dv}{dx}\right)^2 dx = \frac{1}{2L}\int_0^1\left(\frac{dv}{ds}\right)^2 ds \quad \Longleftarrow \quad \frac{dv}{ds} = \left\{\frac{d\mathbf{N}}{ds}\right\}\{\mathbf{q}\}$$

$$\boxed{\Delta^{(e)} = \frac{1}{2}\{\mathbf{q}\}^T[\mathbf{k}_{inc}]\{\mathbf{q}\}}$$

$$[\mathbf{k}_{inc}] = \frac{1}{L}\int_0^1 \{\mathbf{N}'\}\{\mathbf{N}'\}^T ds = \boxed{\frac{1}{30L}\left[\begin{array}{cccc} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{array}\right]\begin{array}{c} v_i \\ \theta_i \\ v_j \\ \theta_j \end{array}}$$

Incremental stiffness matrix (element)

FE Modeling for Buckling

- Potential (work done) by axial load

$$V_{inc} = -\sum_{e=1}^{NE} P^{(e)}(-\Delta^{(e)}) = \sum_{e=1}^{NE} P^{(e)} \frac{1}{2}\{\mathbf{q}\}^{\mathrm{T}}[\mathbf{k}_{inc}^{(e)}]\{\mathbf{q}\}$$

$P^{(e)}$: element axial force

P_r: global axial force

$$V_{inc} = \sum_{e=1}^{NE} P_r \frac{P^{(e)}}{P_r} \frac{1}{2}\{\mathbf{q}\}^{\mathrm{T}}[\mathbf{k}_{inc}^{(e)}]\{\mathbf{q}\} = \frac{1}{2}\{\mathbf{Q}\}^{\mathrm{T}}[P_r\mathbf{K}_{inc}]\{\mathbf{Q}\}$$

$$\boxed{[P_r\mathbf{K}_{inc}] = \text{Assembly}\left\{\sum_{e=1}^{NE} P_r \frac{P^{(e)}}{P_r}[\mathbf{k}_{inc}^{(e)}]\right\}}$$

Global incremental stiffness matrix

- Total potential energy

$$\Pi = \frac{1}{2}\{\mathbf{Q}\}^{\mathrm{T}}[\mathbf{K}]\{\mathbf{Q}\} - \{\mathbf{Q}\}^{\mathrm{T}}\{\mathbf{F}\} + \frac{1}{2}\{\mathbf{Q}\}^{\mathrm{T}}[\mathrm{P}_r\,\mathbf{K}_{inc}]\{\mathbf{Q}\}$$

$$= \frac{1}{2}\{\mathbf{Q}\}^{\mathrm{T}}[\mathbf{K} + P_r\mathbf{K}_{inc}]\{\mathbf{Q}\} - \{\mathbf{Q}\}^{\mathrm{T}}\{\mathbf{F}\}$$

  - [$\mathbf{K}_{inc}$] add a positive definite matrix to the stiffness matrix, thus increasing the strain energy of the system
  - Negative axial force makes the beam softer or more compliant

Eigenvalue Problem for Buckling

- Apply PMPE

$$\frac{d\Pi}{d\{\mathbf{Q}\}} = \mathbf{0} \quad \Rightarrow \quad \underbrace{[\mathbf{K} + P_r\mathbf{K}_{inc}]}_{[\mathbf{K}_T]\text{: total stiffness matrix}}\{\mathbf{Q}\} = \{\mathbf{F}\}$$

  - If the axial force P_r is such that $|\mathbf{K}_T| = 0$, then $\{\mathbf{Q}\}$ is unbounded
  - Since $[\mathbf{K}]$ is positive, $P_r < 0$ (compressive) to make $|\mathbf{K}_T| = 0$
- Critical load (P_{cr}): The negative value of P_r to make $|\mathbf{K}_T| = 0$

$$\boxed{|\mathbf{K} - P_{cr}\mathbf{K}_{inc}| = 0}$$

  - Usually the lowest P_{cr} is of concern
- For calculating P_{cr}, external forces do not matter

$$[\mathbf{K} - P_{cr}\mathbf{K}_{inc}]\{\mathbf{Q}\} = \{\mathbf{0}\}$$

$$[\mathbf{K}]\{\mathbf{Q}\} = P_{cr}[\mathbf{K}_{inc}]\{\mathbf{Q}\}$$

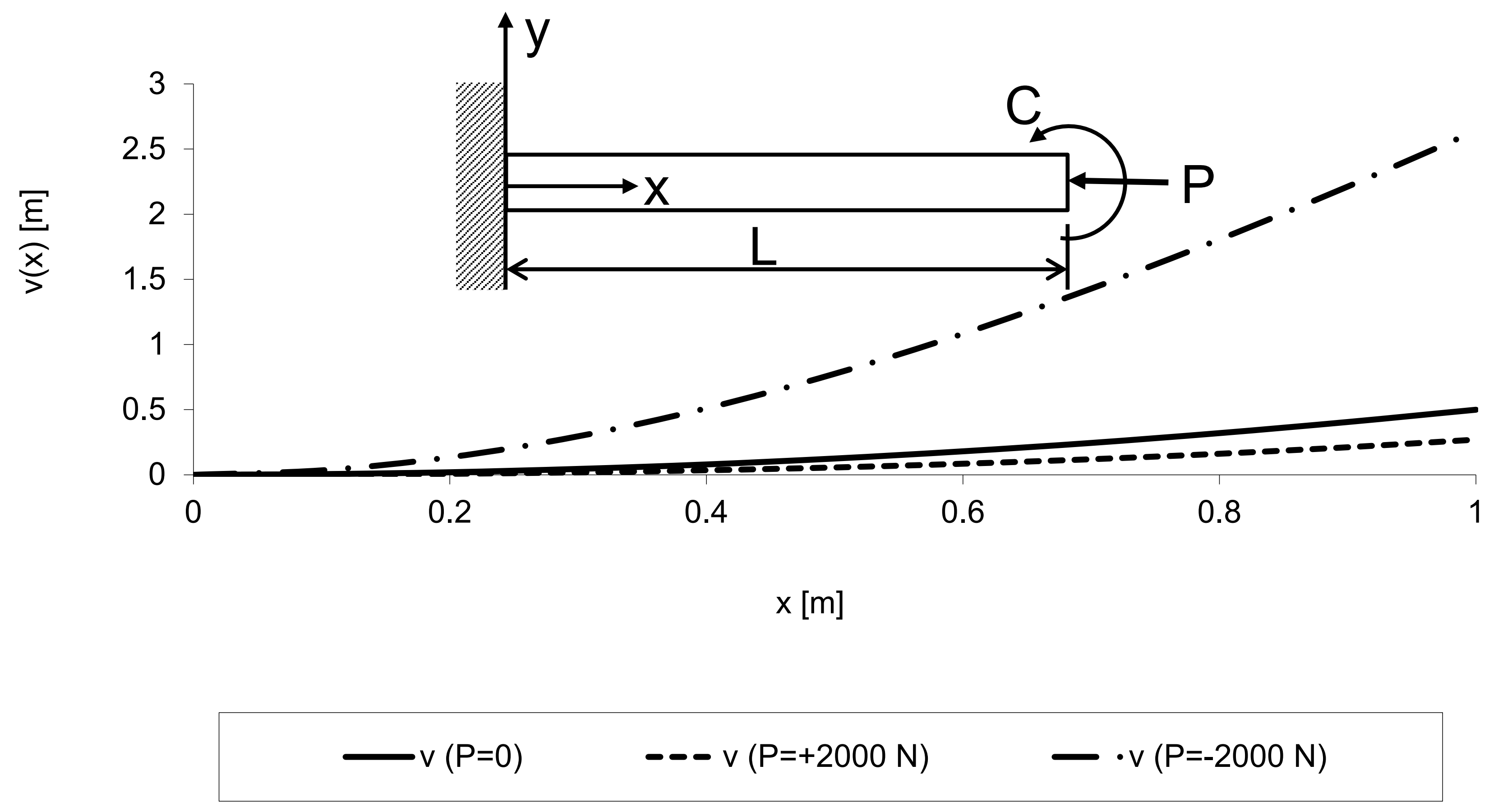
Eigenvalue problem for buckling

$\{\mathbf{Q}\}$: Mode shape of the buckled beam

Cantilever Beam Example

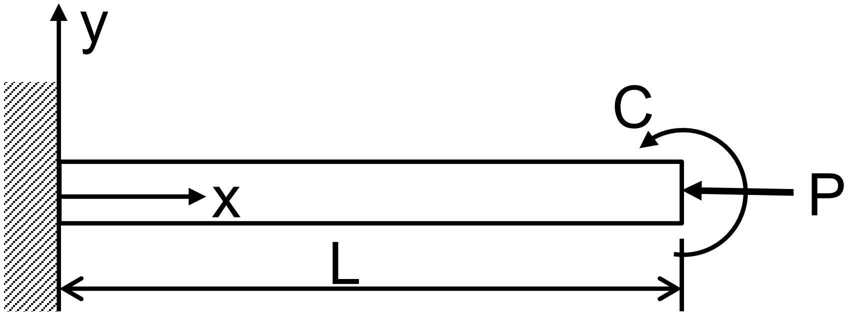
$$[\mathbf{k}^{(e)}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$[\mathbf{k}_{inc}] = \frac{1}{30L}\left[\begin{array}{cccc|c} 36 & 3L & -36 & 3L & v_i \\ 3L & 4L^2 & -3L & -L^2 & \theta_i \\ -36 & -3L & 36 & -3L & v_j \\ 3L & -L^2 & -3L & 4L^2 & \theta_j \end{array}\right]$$



Example: Buckling of a Cantilevered Beam

- Cantilevered beam with L = 1m, EI = 1,000 Nm^2, C = 1,000Nm, calculate P_{cr} using one beam element



- Exact solution:

$$P_{cr1}^{\text{exact}} = \frac{\pi^2 EI}{4L^2} = 2,467\text{N}$$

$$P_{cr2}^{\text{exact}} = \frac{9\pi^2 EI}{4L^2} = 22,207\text{N}$$

$$\lambda_1 = \sqrt{\frac{P_{cr1}}{EI}} = \frac{\pi}{2L}, \quad \lambda_2 = \sqrt{\frac{P_{cr2}}{EI}} = \frac{3\pi}{2L}$$

- Finite element solution
  - Single element, apply BC in element level

$$[\mathbf{K}] = 1000\begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix}\begin{matrix} v_2 \\ \theta_2 \end{matrix} \quad [\mathbf{K}_{inc}] = \frac{1}{30}\begin{bmatrix} 36 & -3 \\ -3 & 4 \end{bmatrix} \quad \{\mathbf{F}\} = \begin{Bmatrix} 0 \\ 1,000 \end{Bmatrix} \quad \{\mathbf{Q}\} = \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix}$$

Example: Buckling of a Cantilevered Beam *cont.*

- Eigenvalue problem

$$[\mathbf{K} - P_{cr}\mathbf{K}_{inc}]\{\mathbf{Q}\} = 0 \quad \Rightarrow \quad \begin{bmatrix} 12000 - 1.2P_{cr} & -6000 + 0.1P_{cr} \\ -6000 + 0.1P_{cr} & 4000 - 0.133P_{cr} \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

  - Let $\beta = P_{cr}/1000$

$$|\mathbf{K} - P_{cr}\mathbf{K}_{inc}| = 0 \quad \Rightarrow \quad (12 - 1.2\beta)(4 - 0.133\beta) - (-6 + 0.1\beta)^2 = 0$$

$$\beta_1 = 2.468, \beta_2 = 32.180 \quad \Rightarrow \quad \boxed{\begin{aligned} P_{cr1} &= 2{,}486\text{N}, P_{cr2} = 32{,}180\text{N} \\ P_{cr1}^{\text{exact}} &= 2{,}467\text{N}, P_{cr2}^{\text{exact}} = 22{,}207\text{N} \end{aligned}}$$

  - Error in the 1st critical load = 1%, 2nd critical load = 45%
  - More elements for accurate higher buckling loads
  - FE critical loads are higher than true values (FE model is stiffer than actual stiffness)

Example: Buckling of a Cantilevered Beam *cont.*

- Mode shape for P_{cr1} = 2,467N

$$[\mathbf{K} - P_{cr1}\mathbf{K}_{inc}]\{\mathbf{Q}_1\} = \begin{bmatrix} 9017 & -5751 \\ -5751 & 3669 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

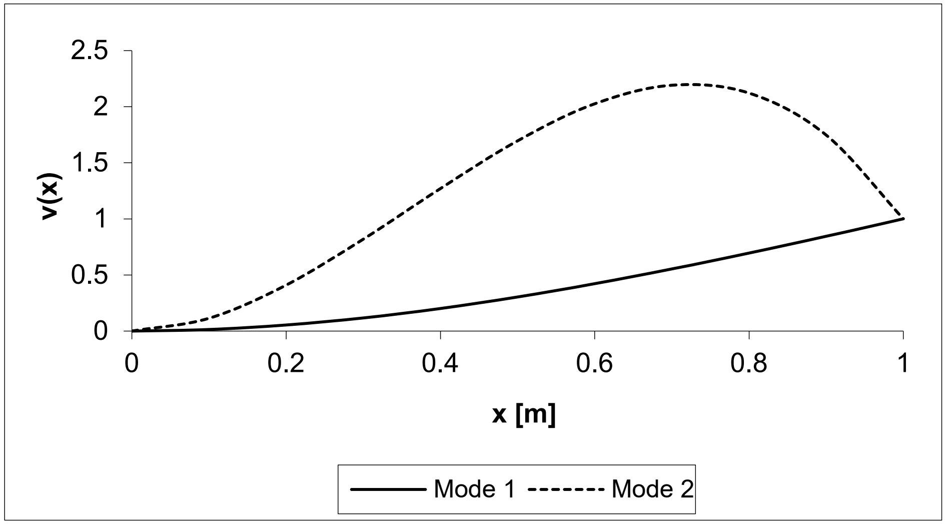
  - Matrix is singular → cannot solve the equation → infinitely many solns
  - Can only get the relation between v_2 and θ_2
  - Choose $v_2 = 1$, $\theta_2 = 1.57$

$$\frac{\theta_2}{v_2} = 1.57$$

$$v(s) = v_2 N_3(s) + \theta_2 N_4(s) = 1.43s^2 - 0.43s^3 \quad \text{(first mode)}$$

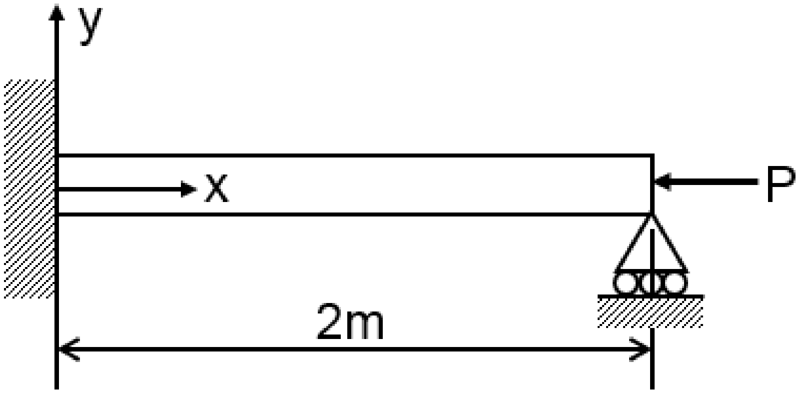
- Mode shape for P_{cr2} = 32,180N ($v_2 = 1$, $\theta_2 = -9.57$)

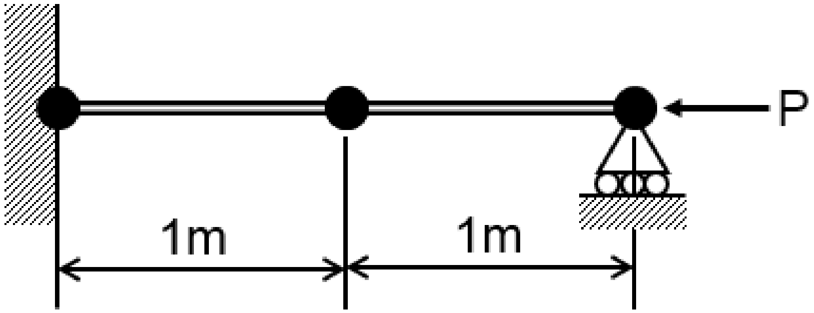
$$v(s) = v_2 N_3(s) + \theta_2 N_4(s) = 12.57s^2 - 11.57s^3 \quad \text{(second mode)}$$



Exercise: Buckling of a Clamped-Hinged Beam

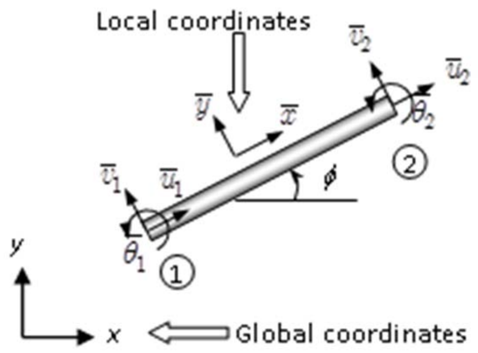
- Calculate the buckling loads and corresponding mode shapes of the beam using: (a) one element and (b) two elements of equal length. Assume beam length 2*L*=2m and *EI*=1,000N-m^2





3.9 BUCKLING OF FRAMES

Incremental Stiffness for Frame



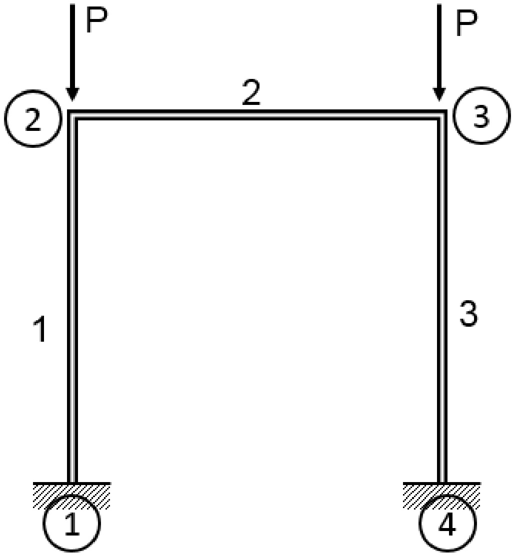
- Frame = Beam + Bar (superposition)
- Expand 4x4 [$\mathbf{k}_{inc}$] to 6x6 in local coordinate

$$
\left[\bar{\mathbf{k}}_{inc}\right] = \frac{1}{30L}\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 36 & 3L & 0 & -36 & 3L \\
0 & 3L & 4L^2 & 0 & -3L & -L^2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -36 & -3L & 0 & 36 & -3L \\
0 & 3L & -L^2 & 0 & -3L & 4L^2
\end{array}\right]
\begin{array}{c}
\bar{u}_1 \\ \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2
\end{array}
$$

- Transform to the global coordinates

$$\boxed{[\mathbf{k}_{inc}] = [\mathbf{T}]^{\mathrm{T}}[\bar{\mathbf{k}}_{inc}][\mathbf{T}]}$$

Example: Buckling of a Portal Frame



- L = 1m, EI=1,000 Nm^2 and EA = 10^{9}N
- Simplification
  - Elem1: u_2, θ_2
  - Elem2: u_2, θ_2, u3, θ_3
  - Elem3: u_3, θ_3
- Element matrices (after BCs)

$$[\mathbf{k}^{(1)}] = 1000\begin{bmatrix} 12 & 6 \\ 6 & 4 \end{bmatrix}\begin{matrix} u_2 \\ \theta_2 \end{matrix}; \quad [\mathbf{k}^{(3)}] = 1000\begin{bmatrix} 12 & 6 \\ 6 & 4 \end{bmatrix}\begin{matrix} u_3 \\ \theta_3 \end{matrix}$$

$$[\mathbf{k}^{(2)}] = 1000\begin{bmatrix} 1\times 10^6 & 0 & -1\times 10^6 & 0 \\ 0 & 4 & 0 & 2 \\ -1\times 10^6 & 0 & 1\times 10^6 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}\begin{matrix} u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{matrix}$$

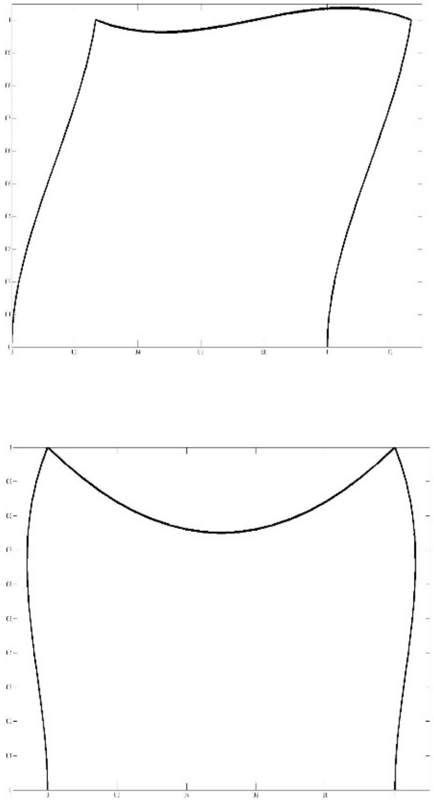
Assembly $\Longrightarrow$ $$[\mathbf{K}] = 1000\begin{bmatrix} 12+10^6 & 6 & -1\times 10^6 & 0 \\ 6 & 8 & 0 & 2 \\ -1\times 10^6 & 0 & 12+10^6 & 6 \\ 0 & 2 & 6 & 8 \end{bmatrix}\begin{matrix} u_2 \\ \theta_2 \\ u_3 \\ \theta_3 \end{matrix}$$

Example: Buckling of a Portal Frame

- Incremental Stiffness
  - Elem 2 does not have axial force → No incremental stiffness

$$[\mathbf{k}_{inc}^{(1)}] = \frac{1}{30}\left[\begin{array}{cc|c} 36 & 3 & u_2 \\ 3 & 4 & \theta_2 \end{array}\right]; [\mathbf{k}_{inc}^{(3)}] = \frac{1}{30}\left[\begin{array}{cc|c} 36 & 3 & u_3 \\ 3 & 4 & \theta_3 \end{array}\right]$$

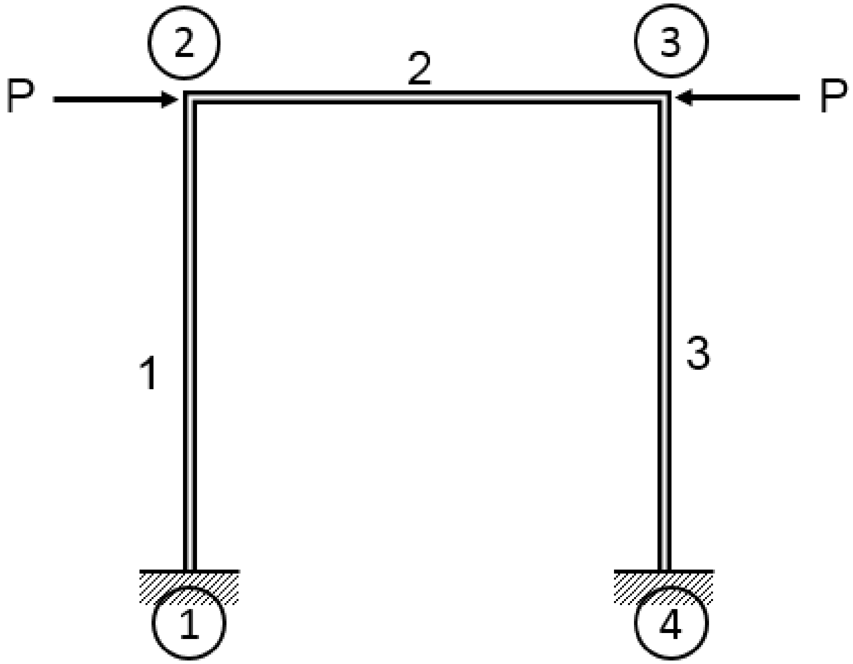
$$[\mathbf{K}_{inc}] = \frac{1}{30}\left[\begin{array}{cccc|c} 36 & 3 & 0 & 0 & u_2 \\ 3 & 4 & 0 & 0 & \theta_2 \\ 0 & 0 & 36 & 3 & u_3 \\ 0 & 0 & 3 & 4 & \theta_3 \end{array}\right]$$



- Solve eigenvalue problem using MATLAB
  - eig(K,K$_{inc}$)
  - P_{cr1}=7,445N and P_{cr2}=45,000N
  - {$\mathbf{Q}_1$}={0.666, -0.388, 0.666, -0.388}
  - {$\mathbf{Q}_2$}={0, -1.94, 0, 1.94}
  - $u_2 = u_3$: axial stiffness in infinitely large
  - Approximate solution: need more elements to represent mode shape

Exercise

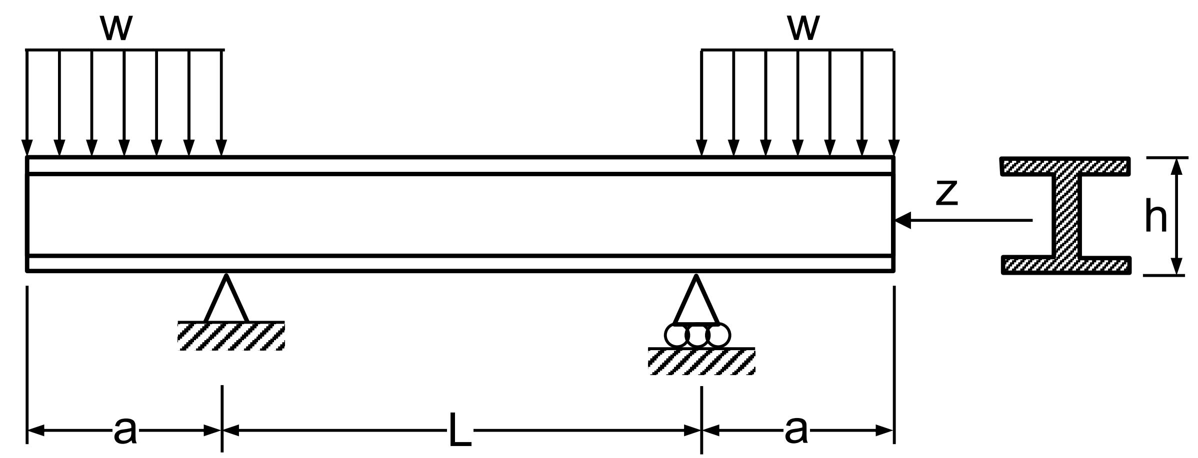
- A pair of loads *P* acts on the horizontal member of the portal frame. Determine the critical load *P* for buckling of the frame. L = 1m, EI=1,000Nm^2 and EA=10^{9}N.

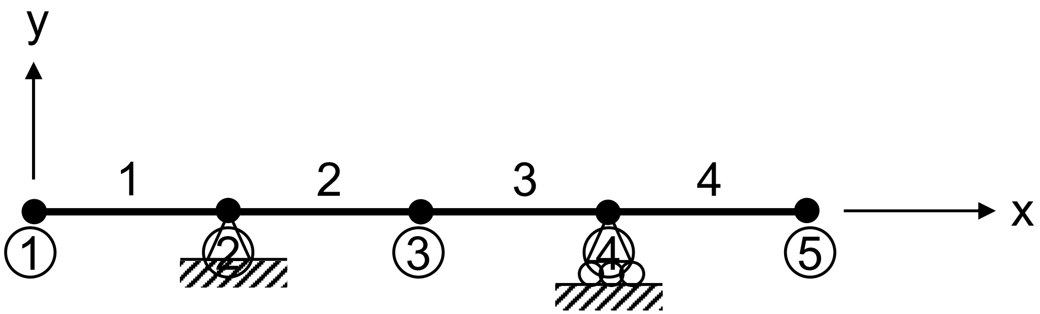


3.10 FE MODELING PRACTICES FOR BEAMS

Stress and Deflection Analysis of a Beam

- Simply supported beam with distributed load at overhangs
- Determine the maximum bending stress σ and the deflection δ at the middle portion
- Standard 30in wide-flange beam, $A = 50.65\text{in}^2$, $I_{zz} = 7892\text{in}^4$, $w = 10{,}000\text{lb/ft}$, $E = 3.0\text{x}10^7\text{psi}$, $L = 20\text{ft}$, $a = 10\text{ft}$, $h = 30\text{in.}$





Stress and Deflection Analysis of a Beam *cont*.

- bending moment in the middle portion (pure bending)

$$M = \frac{wa^2}{2} = -6 \times 10^6 \mathrm{lb} \cdot \mathrm{in}$$

- Stress on the top surface of the middle portion

$$\sigma_{xx} = -\frac{M\frac{h}{2}}{I_z} = -\frac{-6 \times 10^6 \times 15}{7892} = 11{,}404\mathrm{psi}$$

- Deflection: $EI_{zz}y'' = M = \text{constant}$

$$y = a_0 + a_1 x + a_2 x^2 \qquad y(0) = y(240) = 0$$

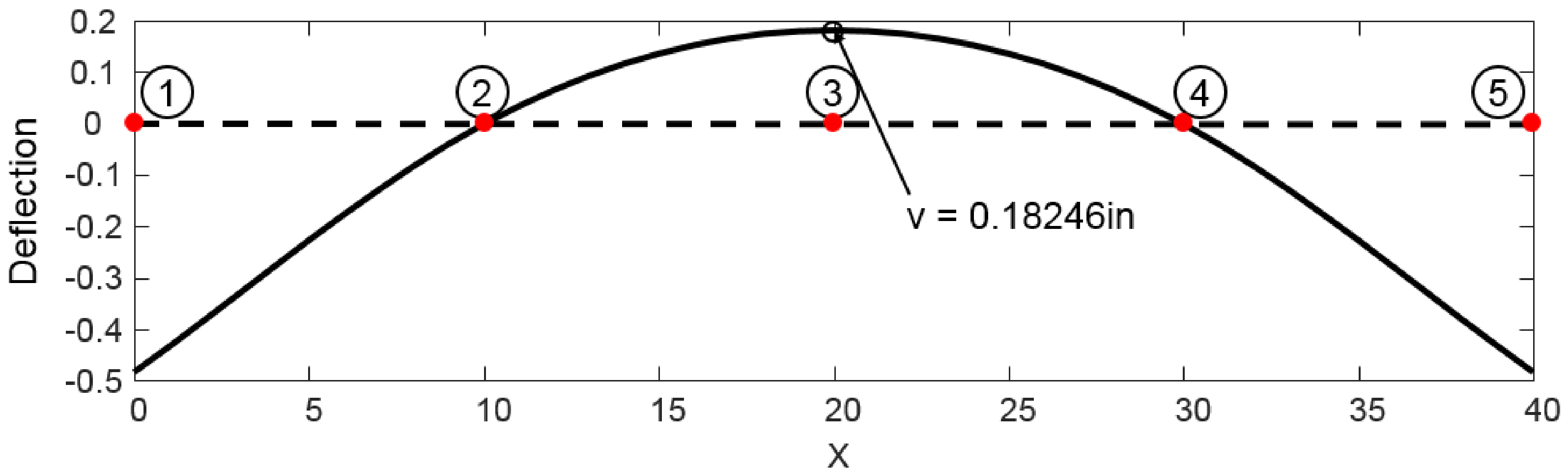
$$y(x) = a_2 x(x - 240), \quad y'' = 2a_2$$

$$a_2 = \frac{M}{2EI_z} = -1.2671 \times 10^{-5}$$

$$y(120) = -1.2671 \times 10^{-5} \times 120 \times (120 - 240) = 0.1825\mathrm{in}$$

Stress and Deflection Analysis of a Beam *cont*.

- Single element for middle section (pure bending), but we will use 2 elements to get deflection at the middle section + 2 elements for overhang



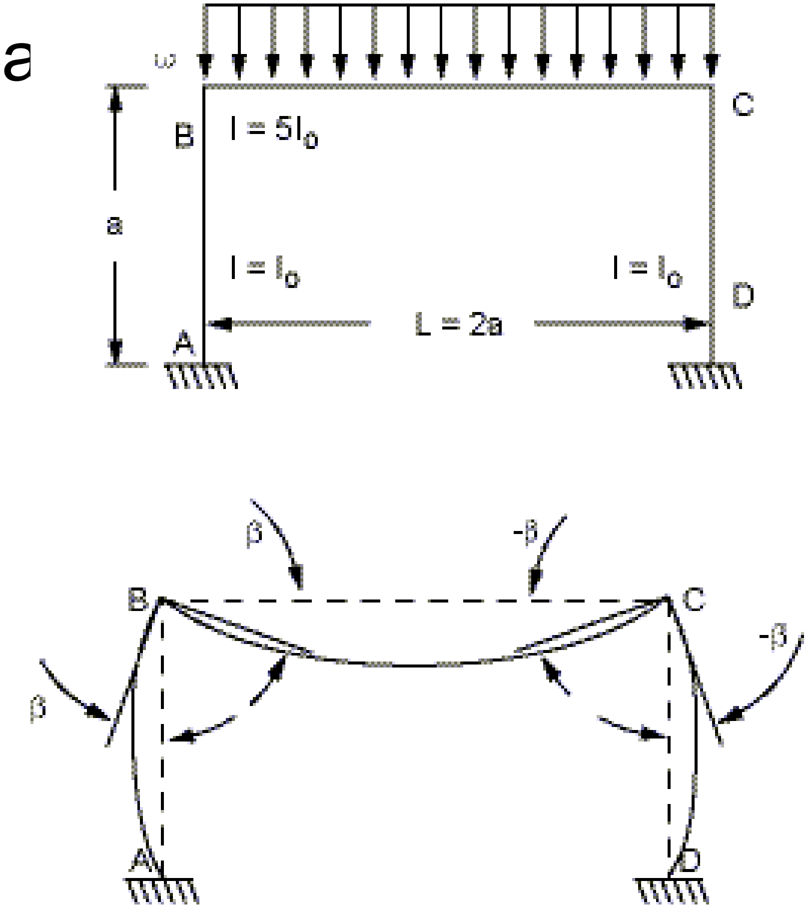
$$M(s) = \frac{EI}{L^2}\lfloor \mathbf{B} \rfloor \{\mathbf{q}\} = -6 \times 10^6 \mathrm{lb} \cdot \mathrm{in}$$

Portal Frame under Symmetric Loading

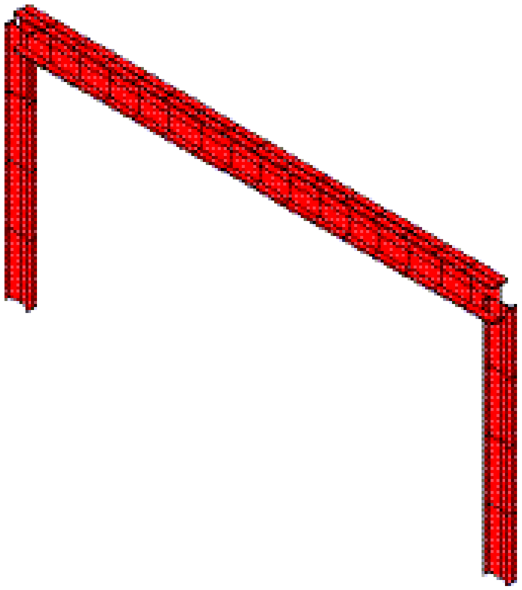
- I-Beam sections with a uniformly distributed load ω = 500lb/in across the span
- determine the maximum rotation and maximum bending moment
- I_{col} = 20,300in^4 a

$$\theta_{\max} = \frac{1}{27} wa^3 \;/\; EI_{col}$$

$$M_{\max} = \frac{19}{54} wa^2$$



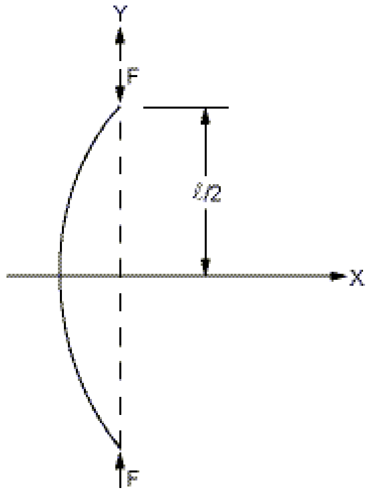
Problem Sketch



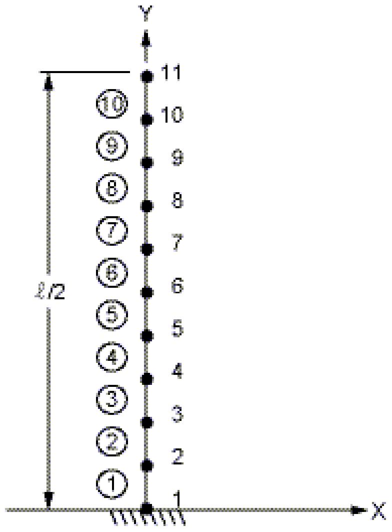
Representative Finite Element Model

Buckling of a Bar with Hinged Ends

- Determine the critical buckling load of an axially loaded long slender bar of length L with hinged ends, as shown in Figure 3.. The bar has a square cross-section with width and height set to 0.5 inches. Determine the critical buckling load of an axially loaded long slender bar of length with hinged ends. The bar has a square cross-section with width and height set to 0.5 inches



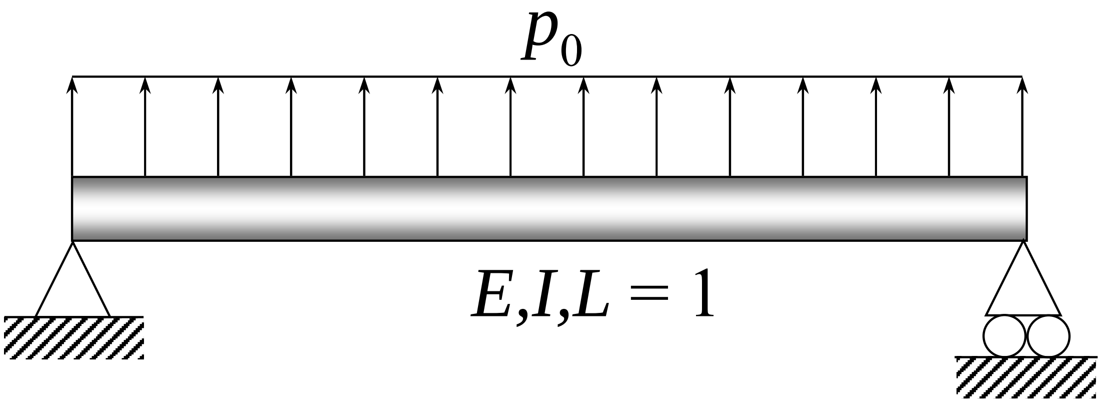
Problem Sketch



Representative Finite Element Model

Exercise

- Calculate the deflection curve v(s), bending moment M(s), and shear force Vy(s) of the simply-supported beam :



$$N_1(s) = 1 - 3s^2 + 2s^3$$
$$N_2(s) = L(s - 2s^2 + s^3)$$
$$N_3(s) = 3s^2 - 2s^3$$
$$N_4(s) = L(-s^2 + s^3)$$

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} pL/2 \\ pL^2/12 \\ pL/2 \\ -pL^2/12 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ C_1 \\ F_2 \\ C_2 \end{Bmatrix}$$