CHAP 4 FINITE ELEMENTS FOR HEAT TRANSFER PROBLEMS

HEAT CONDUCTION ANALYSIS

- Analogy between Stress and Heat Conduction Analysis

Structural problem	Heat transfer problem
Displacement	Temperature (scalar)
Stress/strain	Heat flux (vector)
Displacement B.C.	Fixed temperature B.C.
Surface traction force	Heat flux B.C.
Body force	Internal heat generation
Young's modulus	Thermal conductivity

  – In finite element viewpoint, two problems are identical if a proper interpretation is given.

- More Complex Problems
  – Coupled structural-thermal problems (thermal strain).
  – Radiation problem

THERMAL PROBLEM

- Goals:

$$[\mathbf{K}_T]\{\mathbf{T}\} = \{\mathbf{Q}\}$$

$\{\mathbf{Q}\}$ → Thermal load

$\{\mathbf{T}\}$ → Nodal temperature

$[\mathbf{K}_T]$ → Conductivity matrix

  - Solve for temperature distribution for a given thermal load.
- Boundary Conditions
  - Essential BC: Specified temperature
  - Natural BC: Specified heat flux

4.2. FOURIER HEAT CONDUCTION EQUATION

STEADY-STATE HEAT TRANSFER PROBLEM

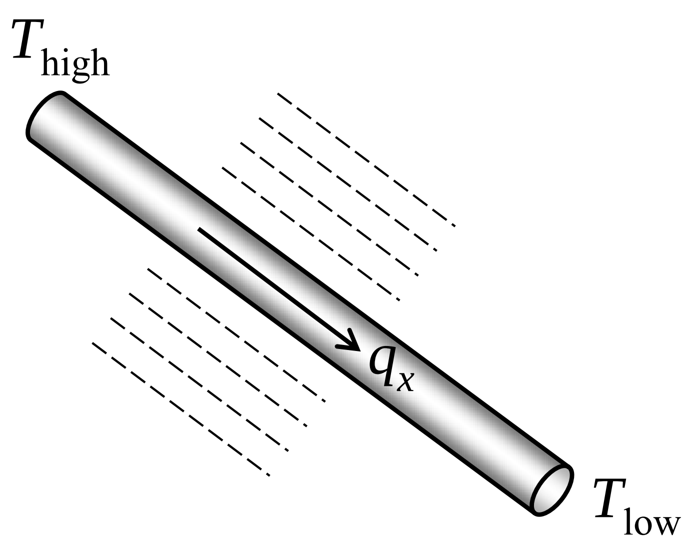
- Fourier Heat Conduction Equation:
  - Heat flow from high temperature to low temperature

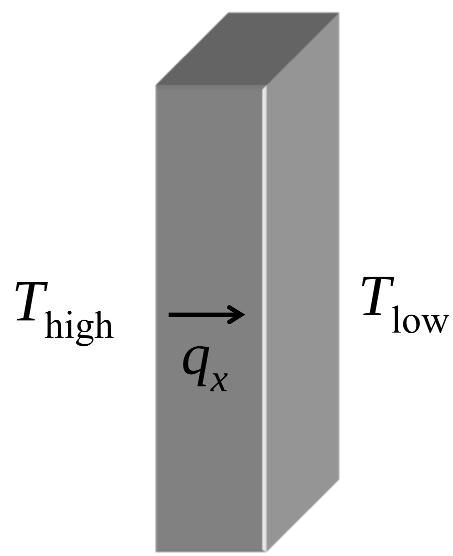
$$q_x = -kA\frac{\mathrm{d}T}{\mathrm{d}x}$$

k → Thermal conductivity (W/m/°C)

q_x → Heat flux (Watts)

- Examples of 1D heat conduction problems



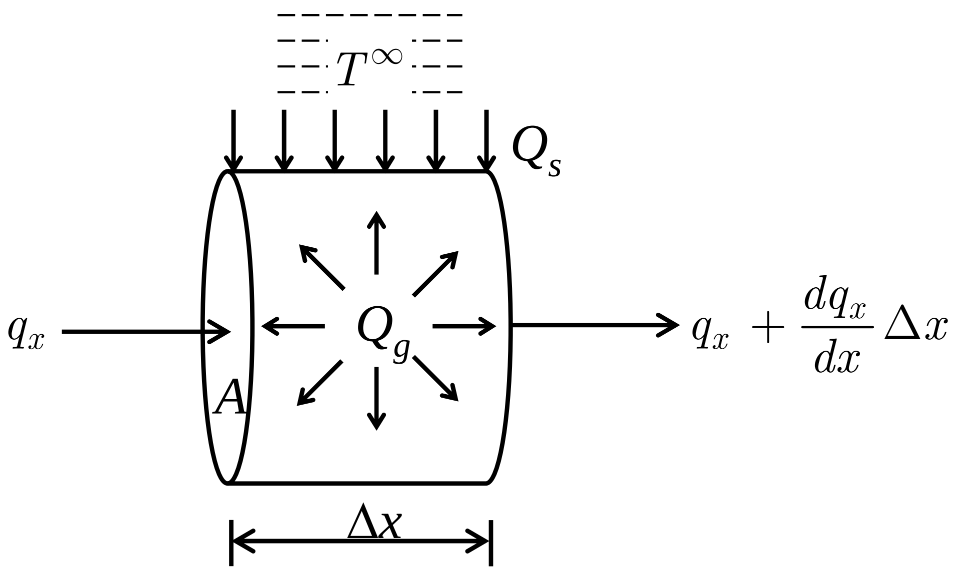


GOVERNING DIFFERENTIAL EQUATION

- Conservation of Energy
  - Energy In + Energy Generated = Energy Out + Energy Increase

$$E_{in} + E_{generated} = E_{out} + \Delta U$$

- Two modes of heat transfer through the boundary
  - Prescribed surface heat flow Q_s per unit area
  - Convective heat transfer $Q_h = h(T^\infty - T)$
  - h: convection coefficient (W/m^2/°C)



GOVERNING DIFFERENTIAL EQUATION cont.

- Conservation of Energy at Steady State
  - No change in internal energy ($\Delta U = 0$)

$$\underbrace{q_x + Q_s P \Delta x + h\left(T^{\infty} - T\right) P \Delta x}_{E_{\text{in}}} + \underbrace{Q_g A \Delta x}_{E_{\text{gen}}} = \underbrace{\left(q_x + \frac{\mathrm{d}q_x}{\mathrm{d}x}\Delta x\right)}_{E_{\text{out}}}$$

  - P: perimeter of the cross-section

$$\frac{\mathrm{d}q_x}{\mathrm{d}x} = Q_g A + hP\left(T^{\infty} - T\right) + Q_s P, \quad 0 \le x \le L$$

- Apply Fourier Law

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right) + Q_g A + hP\left(T^{\infty} - T\right) + Q_s P = 0,} \quad 0 \le x \le L$$

  - Rate of change of heat flux is equal to the sum of heat generated and heat transferred

GOVERNING DIFFERENTIAL EQUATION cont.

- Boundary conditions
  - Temperature at the boundary is prescribed (essential BC)
  - Heat flux is prescribed (natural BC)
  - Example: essential BC at x = 0, and natural BC at x = L:

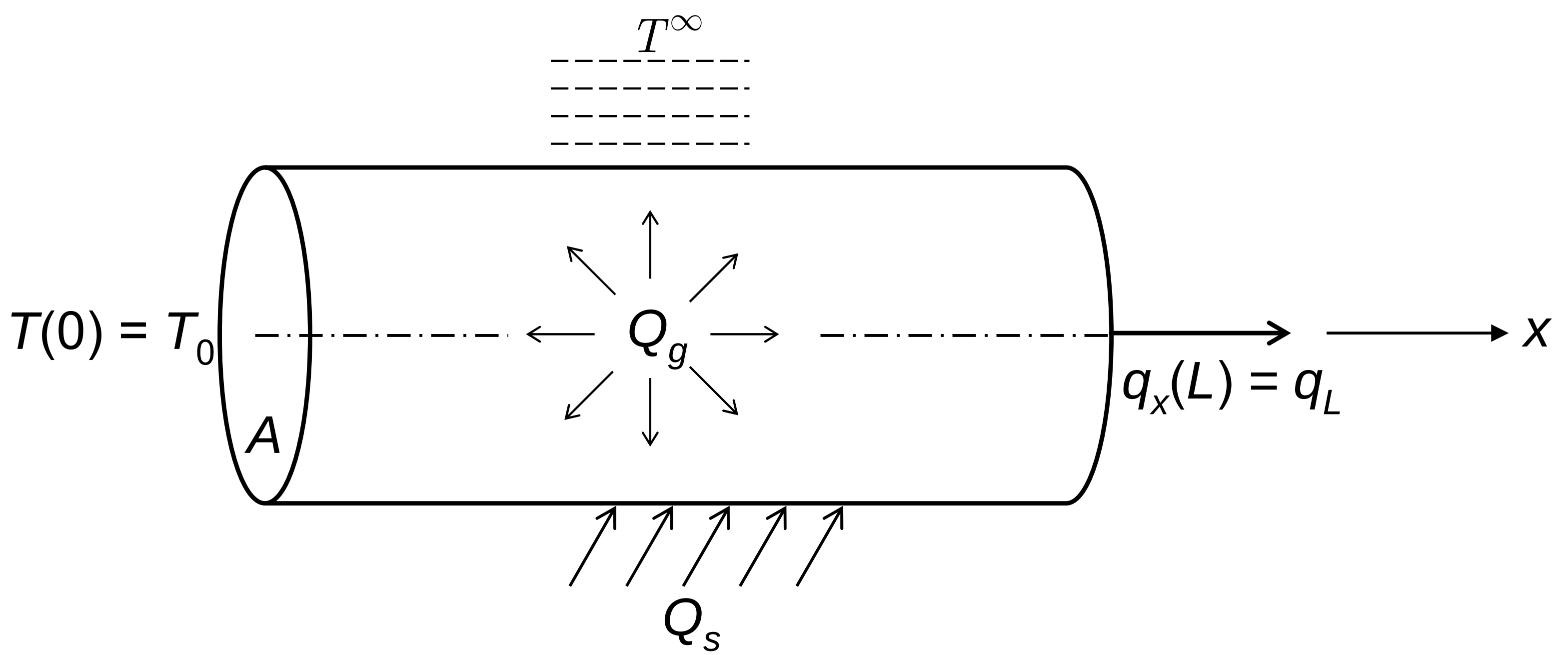
$$\begin{cases} T(0) = T_0 \\ kA\left.\dfrac{dT}{dx}\right|_{x=L} = q_L \end{cases}$$

4.3. FINITE ELEMENT ANALYSIS – DIRECT METHOD

Heat Conduction in a Long Wire

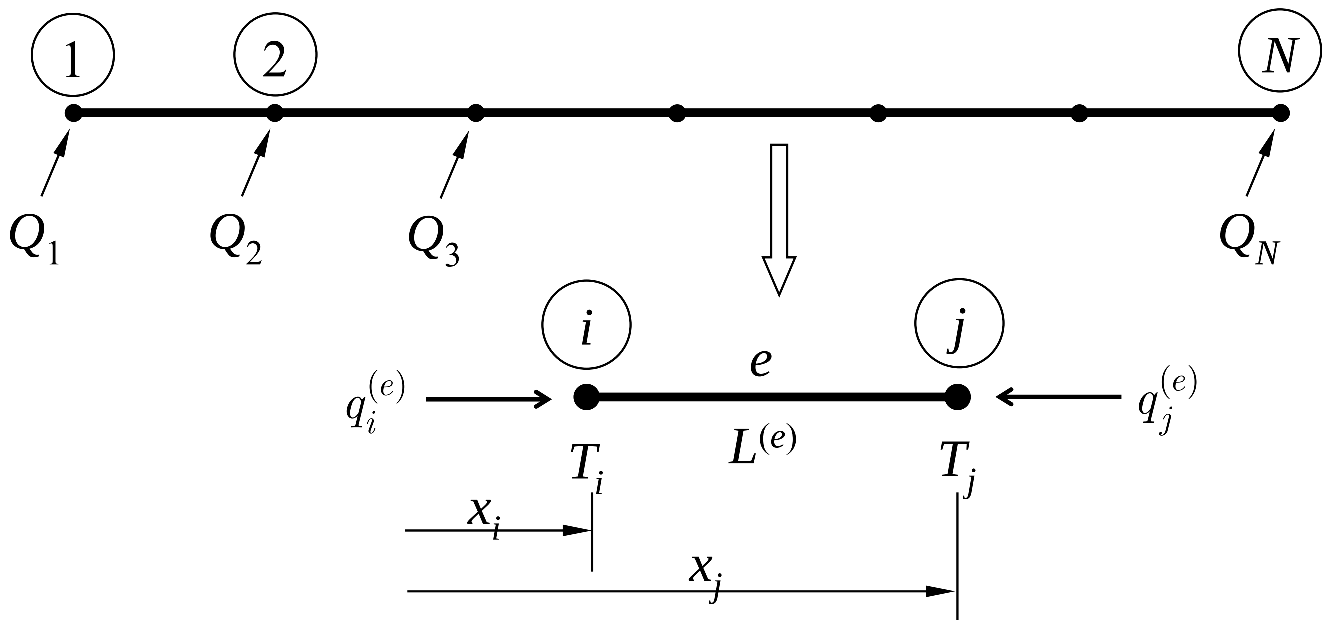
- Similar to direct method in 1D bar
- No need to work with differential equation
- Use conservation of energy

- Heat transfer on the boundary
- Internal heat generation
- No radiation



DIRECT METHOD

- Follow the same procedure with 1D bar element
  - No need to use differential equation
- Element conduction equation
  - Heat can enter the system only through the nodes
  - Q_i: heat enters at node *i* (Watts)
  - Divide the solid into a number of elements
  - Each element has two nodes and two DOFs (T_i and T_j)
  - For each element, heat entering the element is positive



ELEMENT EQUATION

- Fourier law of heat conduction

$$q_i^{(e)} = -kA\frac{\mathrm{d}T}{\mathrm{d}x} = -kA\frac{\left(T_j - T_i\right)}{L^{(e)}}$$

- From the conservation of energy for the element

$$q_i^{(e)} + q_j^{(e)} = 0 \quad \Longrightarrow \quad q_j^{(e)} = +kA\frac{(T_j - T_i)}{L^{(e)}}$$

- Combine the two equation

$$\begin{Bmatrix} q_i^{(e)} \\ q_j^{(e)} \end{Bmatrix} = \frac{kA}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$$

Element conductance matrix

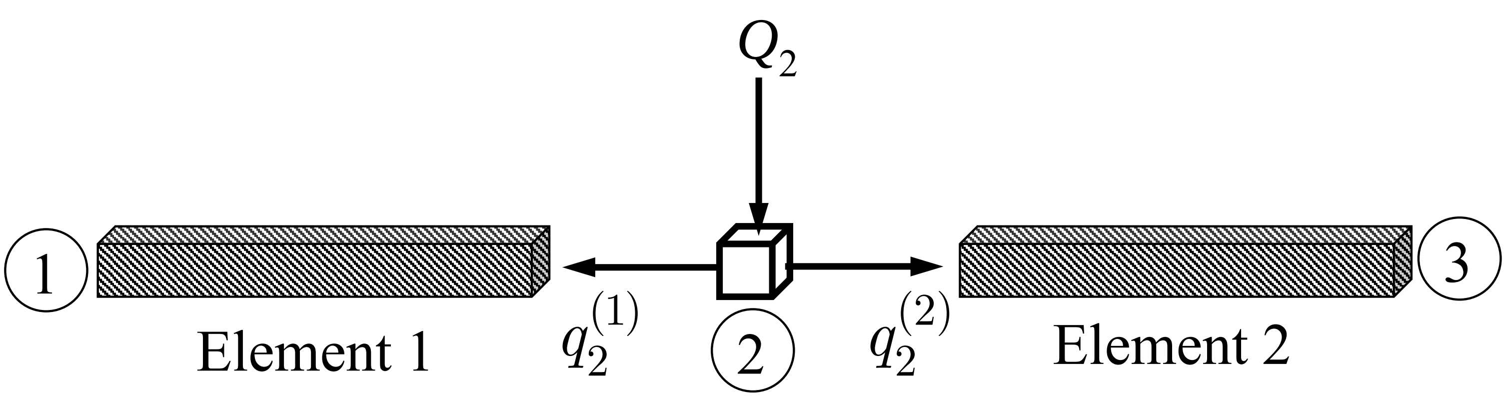
  – Similar to 1D bar element (k = E, T = u, q = f)

ASSEMBLY

- Assembly using heat conservation at nodes
  - Remember that heat flow into the element is positive
  - Equilibrium of heat flow:

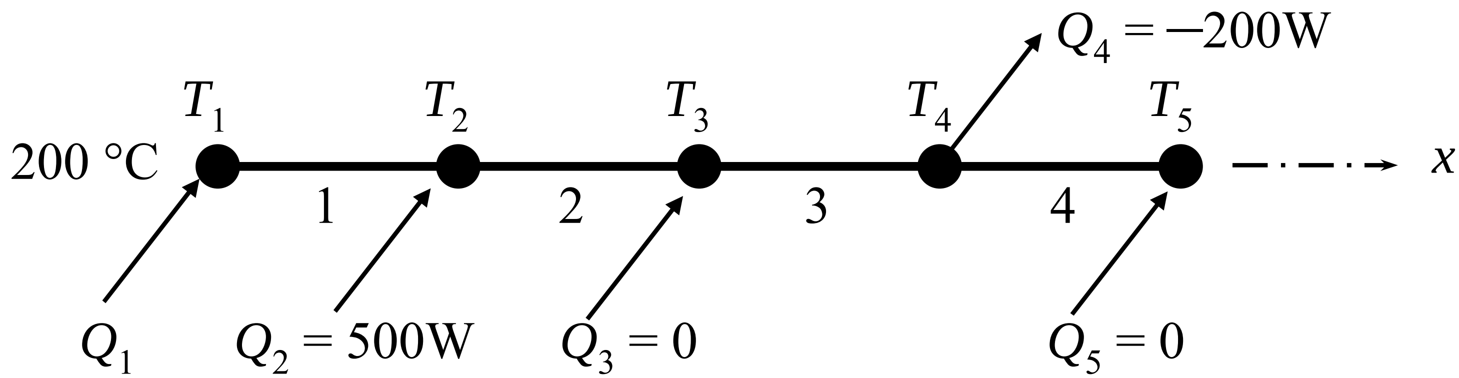
$$Q_i = \sum_{e=1}^{N_i} q_i^{(e)} \quad \Longrightarrow \quad \underset{(N\times N)}{[\mathbf{K}_T]} \begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{Bmatrix}$$

  - Same assembly procedure with 1D bar elements
- Applying BC
  - Striking-the-rows works, but not striking-the-columns because prescribed temperatures are not usually zero



EXAMPLE

- Calculate nodal temperatures of four elements
  - A = 1m^2, L = 1m, k = 10W/m/°C



- Element conduction equation

$$10\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} \end{Bmatrix} \qquad 10\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} q_2^{(2)} \\ q_3^{(2)} \end{Bmatrix}$$

$$10\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} q_3^{(3)} \\ q_4^{(3)} \end{Bmatrix} \qquad 10\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} q_4^{(4)} \\ q_5^{(4)} \end{Bmatrix}$$

EXAMPLE cont.

- Assembly

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} = \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} + q_2^{(2)} \\ q_3^{(2)} + q_3^{(3)} \\ q_4^{(3)} + q_4^{(4)} \\ q_5^{(4)} \end{Bmatrix} = 10 \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix}$$

- Boundary conditions (T_1 = 200 ^{o}C, Q_1 is unknown)

$$10 \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 200 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ 500 \\ 0 \\ -200 \\ 0 \end{Bmatrix}$$

EXAMPLE cont.

- Boundary conditions
  - Strike the first row

$$10\begin{bmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 200 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 500 \\ 0 \\ -200 \\ 0 \end{Bmatrix}$$

  - Instead of striking the first column, multiply the first column with T_1 = 200 °C and move to RHS

$$10\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 500 \\ 0 \\ -200 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 2000 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

  - Now, the global matrix is positive-definite and can be solved for nodal temperatures

EXAMPLE cont.

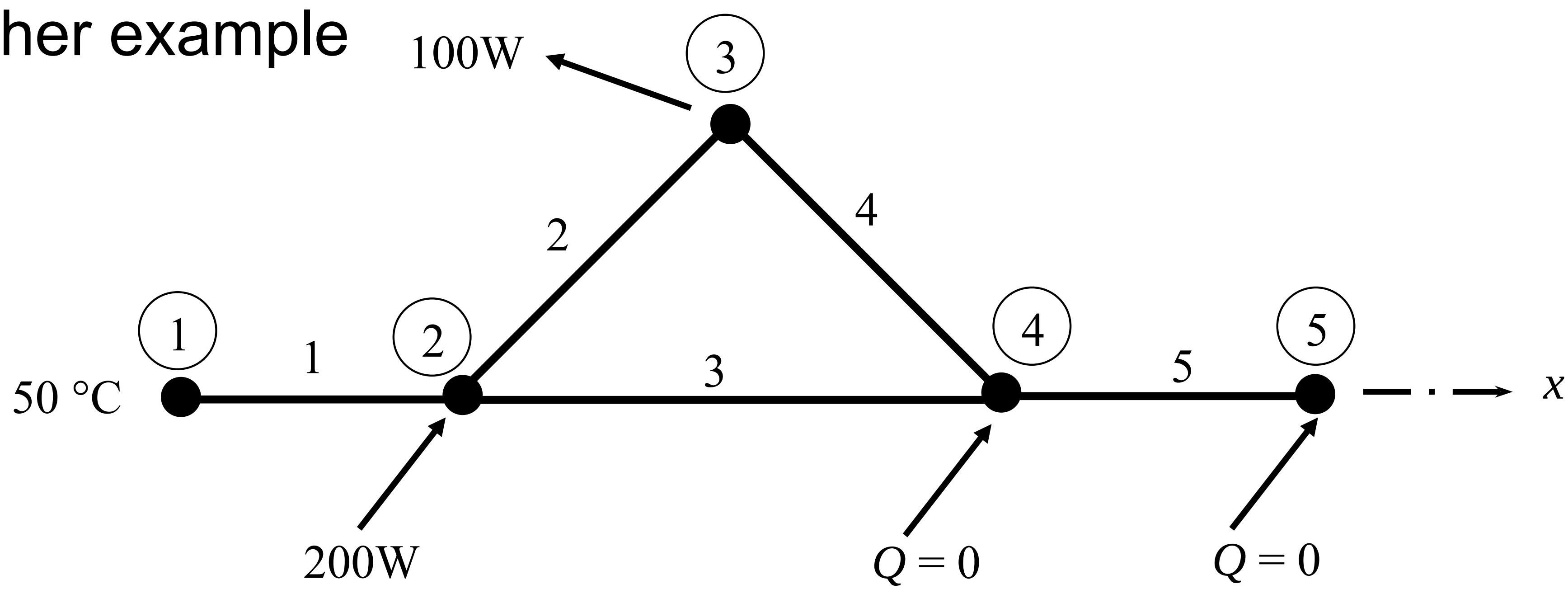
- Nodal temperatures

$$\{\mathbf{T}\}^{\mathrm{T}} = \{200 \quad 230 \quad 210 \quad 190 \quad 190\}\,^{\circ}\mathrm{C}$$

- How much heat input is required to maintain T_1 = 200°C?
  - Use the deleted first row with known nodal temperatures

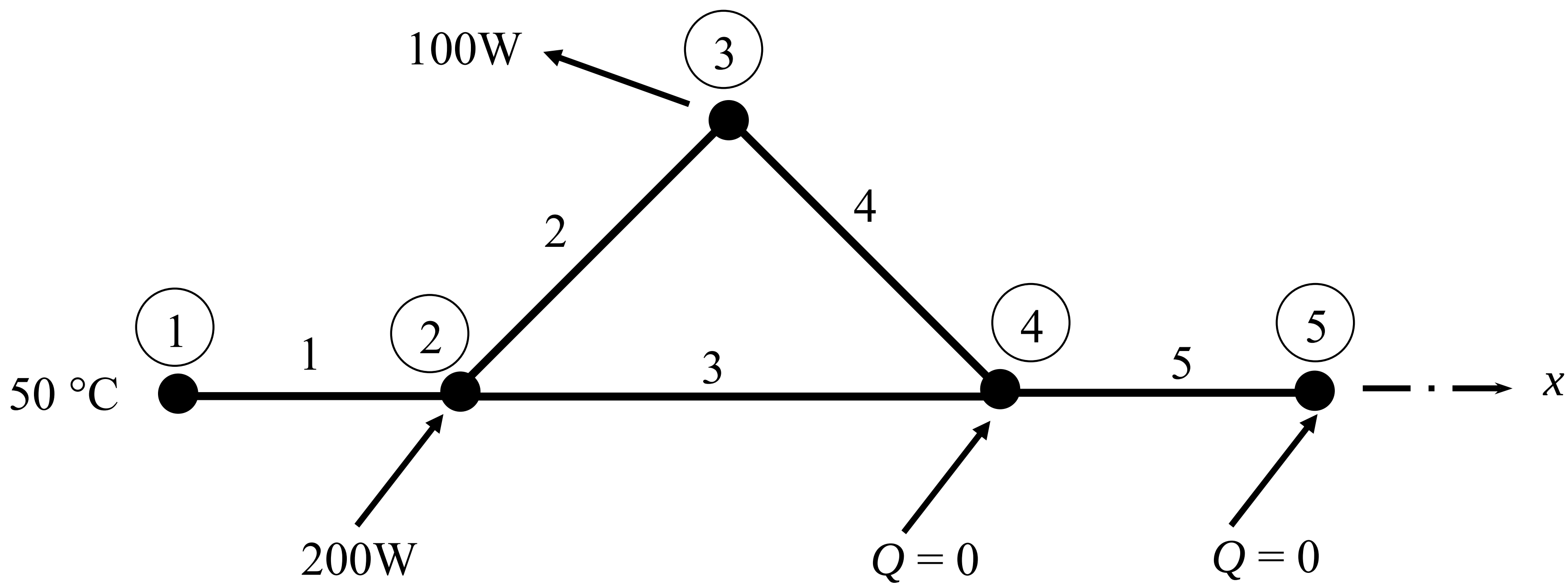
$$Q_1 = 10T_1 - 10T_2 + 0T_3 + 0T_4 + 0T_5 = -300\,\mathrm{W}$$

- Other example



Exercise

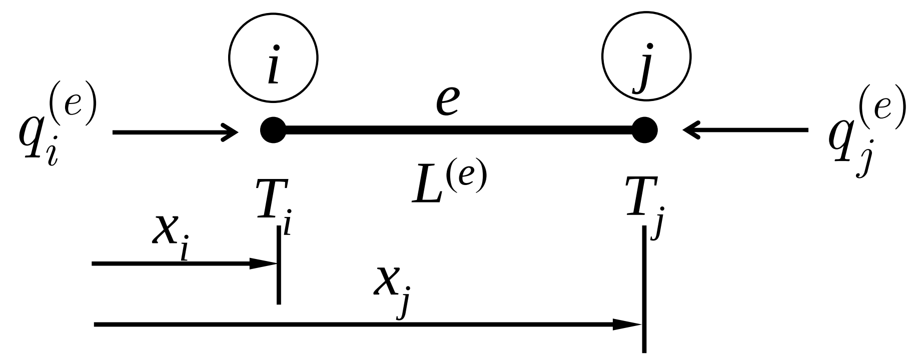
- Calculate nodal temperatures of five elements in the figure
  - A = 1m^2, L = 1m, k = 10W/m/°C



4.4. GALERKIN METHOD FOR HEAT CONDUCTION

GALERKIN METHOD FOR HEAT CONDUCTION

- Direct method is limited for nodal heat input
- Need more advanced method for heat generation and convection heat transfer
- Galerkin method in Chapter 2 can be used for this purpose
- Consider element (e)



- Interpolation

$$\tilde{T}(x) = T_i N_i(x) + T_j N_j(x)$$

$$N_i(x) = \left(1 - \frac{x - x_i}{L^{(e)}}\right), \quad N_j(x) = \frac{x - x_i}{L^{(e)}}$$

$$\tilde{T}(x) = \lfloor \mathbf{N} \rfloor \{\mathbf{T}\} = \lfloor N_i(x) \quad N_j(x) \rfloor \begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$$

Temperature varies linearly

- Heat flux

$$\frac{d\tilde{T}}{dx} = \left\lfloor -\frac{1}{L^{(e)}} \quad \frac{1}{L^{(e)}} \right\rfloor \{\mathbf{T}\} = \lfloor \mathbf{B} \rfloor \{\mathbf{T}\}$$

Heat flow is constant

GALERKIN METHOD cont.

- Differential equation with heat generation

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) + Q_g A = 0, \quad 0 \le x \le L$$

- Substitute approximate solution

$$\frac{d}{dx}\left(kA\frac{d\tilde{T}}{dx}\right) + AQ_g = R(x) \longrightarrow \text{Residual}$$

- Integrate the residual with $N_i(x)$ as a weight

$$\int_{x_i}^{x_j}\left(\frac{d}{dx}\left(kA\frac{d\tilde{T}}{dx}\right) + AQ_g\right)N_i(x)dx = 0$$

- Integrate by parts

$$kA\frac{d\tilde{T}}{dx}N_i(x)\Big|_{x_i}^{x_j} - \int_{x_i}^{x_j} kA\frac{d\tilde{T}}{dx}\frac{dN_i}{dx}dx = -\int_{x_i}^{x_j} AQ_g N_i(x)dx$$

GALERKIN METHOD cont.

- Substitute interpolation relation

$$\int_{x_i}^{x_j} kA\left(T_i\frac{\mathrm{d}N_i}{\mathrm{d}x}+T_j\frac{\mathrm{d}N_j}{\mathrm{d}x}\right)\frac{\mathrm{d}N_i}{\mathrm{d}x}\mathrm{d}x=\int_{x_i}^{x_j} AQ_gN_i(x)\mathrm{d}x-q(x_j)N_i(x_j)+q(x_i)N_i(x_i)$$

- Perform integration

$$\frac{kA}{L^{(e)}}(T_i-T_j)=Q_i^{(e)}+q_i^{(e)} \qquad Q_i^{(e)}=\int_{x_i}^{x_j} AQ_gN_i(x)dx$$

- Repeat with $N_j(x)$ as a weight

$$\frac{kA}{L^{(e)}}(T_j-T_i)=Q_j^{(e)}+q_j^{(e)} \qquad Q_j^{(e)}=\int_{x_i}^{x_j} AQ_gN_j(x)dx$$

GALERKIN METHOD cont.

- Combine the two equations

$$\frac{kA}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix} = \begin{Bmatrix} Q_i^{(e)} + q_i^{(e)} \\ Q_j^{(e)} + q_j^{(e)} \end{Bmatrix}$$

$$[\mathbf{k}_\mathrm{T}^{(e)}]\{\mathbf{T}\} = \{\mathbf{Q}^{(e)}\} + \{\mathbf{q}^{(e)}\}$$

Similar to 1D bar element

  - $\{\mathbf{Q}^{(e)}\}$: thermal load corresponding to the heat source
  - $\{\mathbf{q}^{(e)}\}$: vector of nodal heat flows across the cross-section

- Uniform heat source

$$\{\mathbf{Q}^{(e)}\} = \int_{x_i}^{x_j} AQ_g \begin{bmatrix} N_i(x) \\ N_j(x) \end{bmatrix} \mathrm{d}x = \frac{AQ_g L^{(e)}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

  - Equally divided to the two nodes

- **Temperature varies linearly in the element, and the heat flux is constant**

EXAMPLE

- Heat chamber

Wall temperature = 200 °C

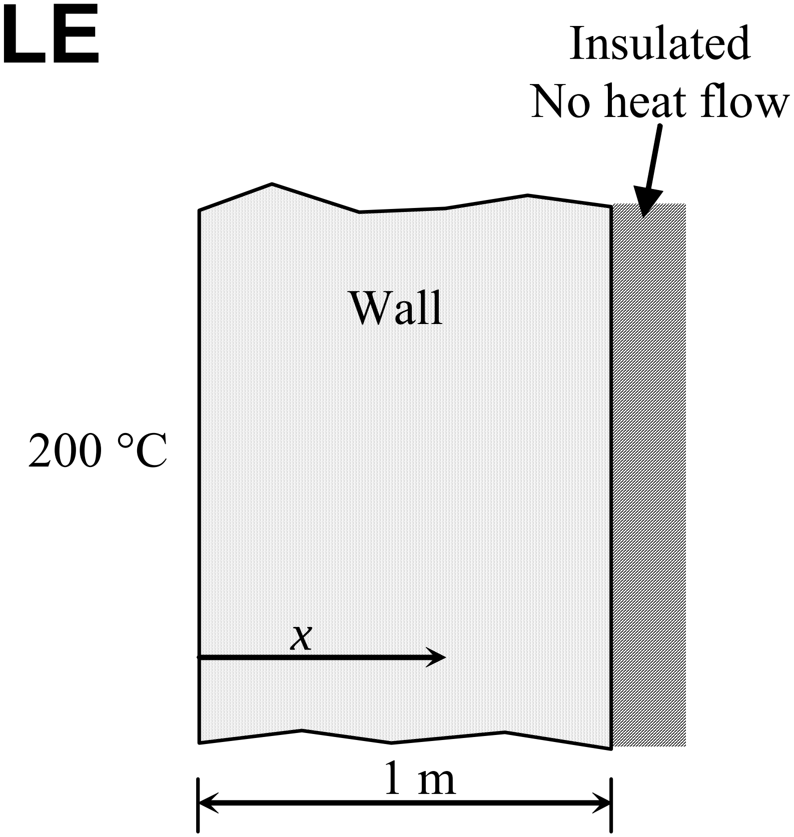
Uniform heat source inside the wall Q = 400 W/m³.

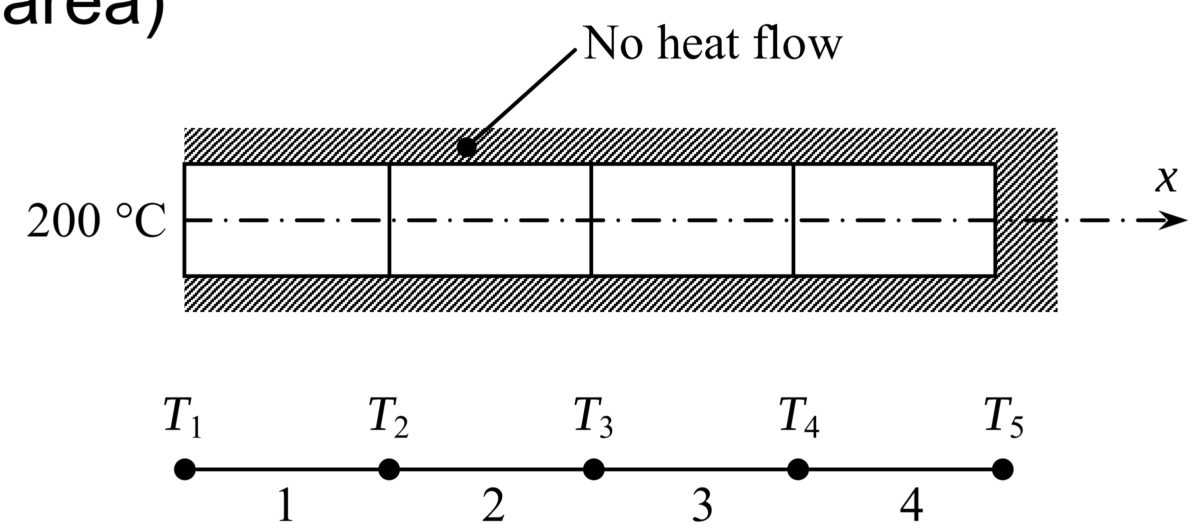
Thermal conductivity of the wall is k = 25 W/m·°C.

Use four elements through the thickness (unit area)

Boundary Condition:

$T_1 = 200$, $q_{x=1} = 0$.





EXAMPLE *cont.*

- Element Matrix Equation
  - All elements are identical

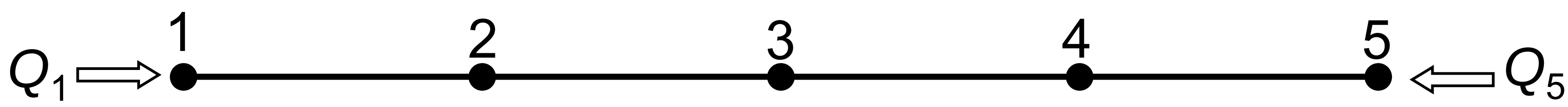
$$100\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 50 \\ 50 \end{Bmatrix} + \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} \end{Bmatrix}$$

  - Assembly

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} = \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} + q_2^{(2)} \\ q_3^{(2)} + q_3^{(3)} \\ q_4^{(3)} + q_4^{(4)} \\ q_5^{(4)} \end{Bmatrix} = 100\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} - \begin{Bmatrix} 50 \\ 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix}$$

EXAMPLE *cont.*

- Boundary Conditions
  - At node 1, the temperature is given (T_1 = 200). Thus, the heat flux at node 1 (Q_1) should be unknown.
  - At node 5, the insulation condition required that the heat flux (Q_5) should be zero. Thus, the temperature at node 5 should be unknown.
  - At nodes 2 – 4, the temperature is unknown (T_2, T_3, T_4). Thus the heat flux should be known.



$$100\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 200 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 50+Q_1 \\ 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix}$$

EXAMPLE *cont.*

- Imposing Boundary Conditions
  - Remove first row because it contains unknown Q_1.
  - Cannot remove first column because T_1 is not zero.

$$100\begin{bmatrix} -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 200 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix}$$

  - Instead, move the first column to the right.

$$100(-1\times 200 + 2\times T_2 - 1\times T_3) = 100$$

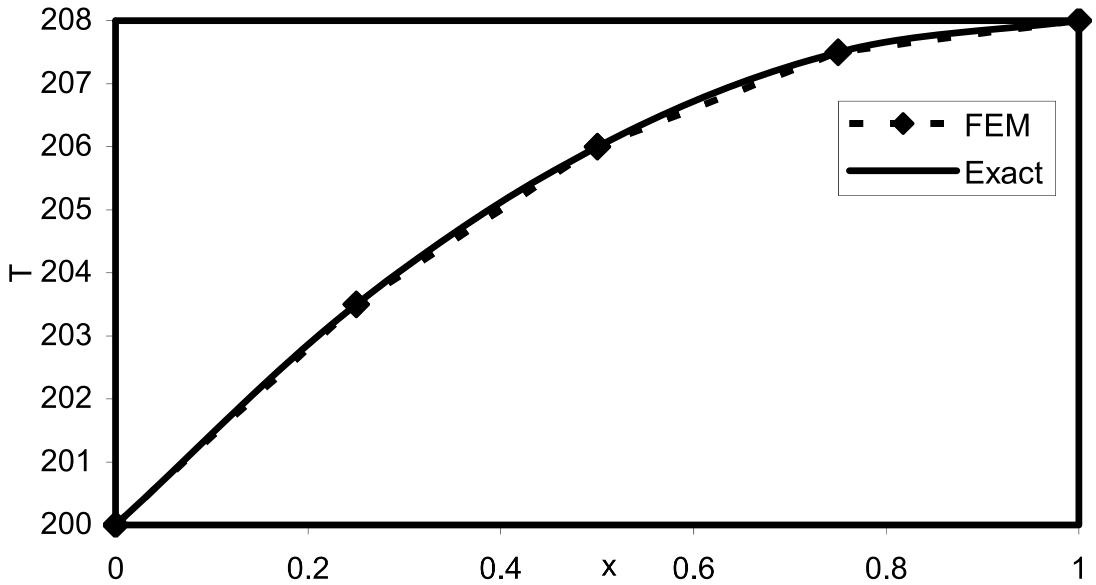
$$100(2\times T_2 - 1\times T_3) = 100 + 20000$$

$$100\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix} + \begin{Bmatrix} 20000 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 20100 \\ 100 \\ 100 \\ 50 \end{Bmatrix}$$

EXAMPLE *cont.*

- Solution

$$T_1 = 200\ °C,\ T_2 = 203.5\ °C,\ T_3 = 206\ °C,\ T_4 = 207.5\ °C,\ T_5 = 208\ °C$$



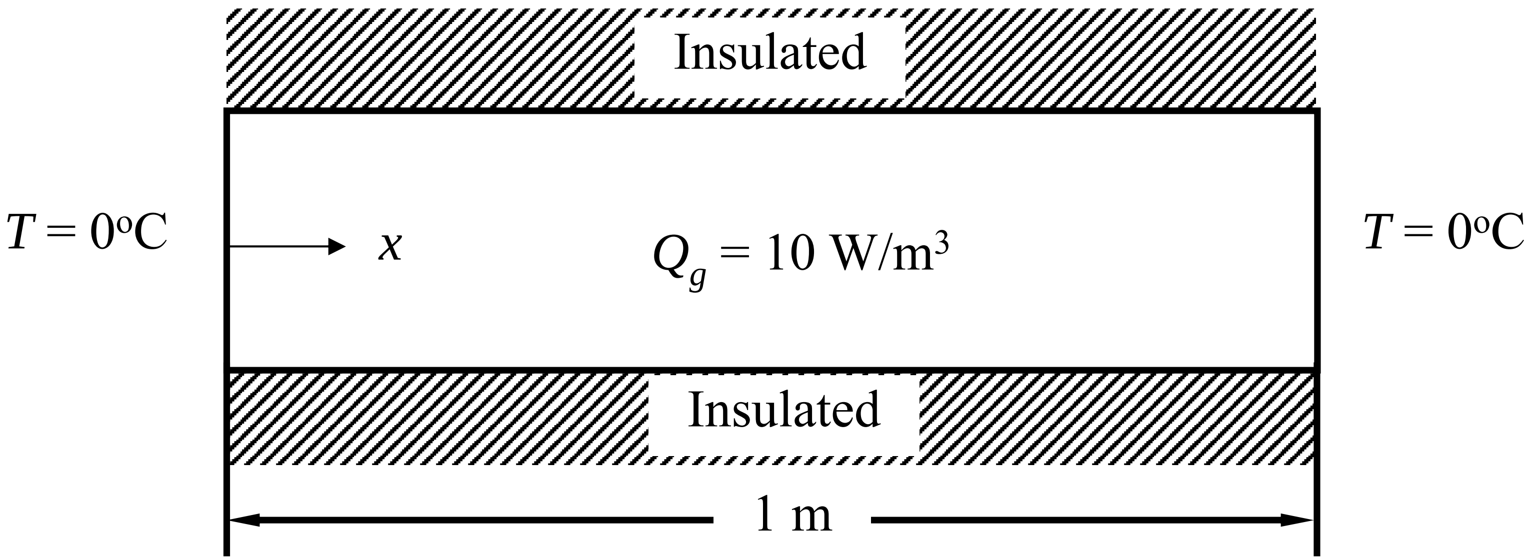
- Discussion
  - In order to maintain 200 degree at node 1, we need to remove heat

$$Q_1 + 50 = 100T_1 - 100T_2 = -350$$

$$\Rightarrow Q_1 = -400\ W$$

Exercise

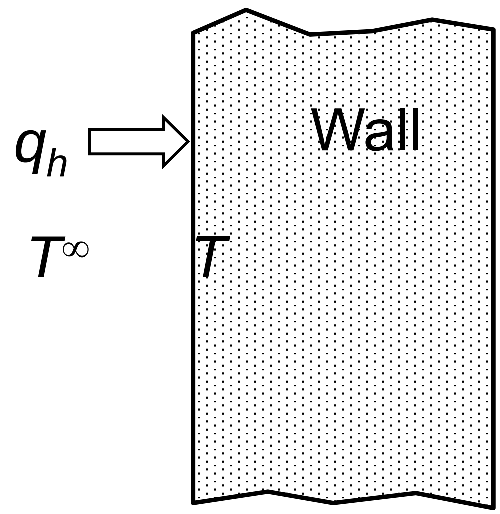
- Consider a heat conduction problem described in the figure. Inside of the domain, heat is generated from a uniform heat source Q_g = 10 W/m^3, and the conductivity of the domain is k = 0.1 W/m/^{o}C. The cross-sectional area A = 1 m^2. When the temperatures at both ends are fixed at 0^{o}C, calculate the temperature distribution using (a) two equal-length elements and (b) three equal-length elements. Plot the temperature distribution along with the exact temperature distribution



4.5. CONVECTION BOUNDARY CONDITION

CONVECTION BC

- Convection Boundary Condition
  - Happens when a structure is surrounded by fluid
  - Does not exist in structural problems
  - BC includes unknown temperature (mixed BC)



$$q_h = hS(T^\infty - T)$$

T^∞: Fluid Temperature

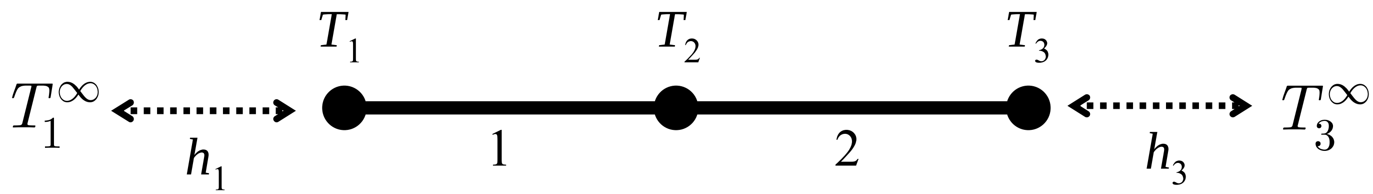
h: Convection Coefficient

  - Heat flow is not prescribed. Rather, it is a function of temperature on the boundary, which is unknown
- 1D Finite Element
  - When both Nodes 1 and 2 are convection boundary

$$\begin{cases} q_1 = hAT_1^\infty - hAT_1 \\ q_2 = hAT_2^\infty - hAT_2 \end{cases}$$

EXAMPLE (CONVECTION ON THE BOUNDARY)

- Element equation



$$\frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} \end{Bmatrix} \qquad \frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} q_2^{(2)} \\ q_3^{(2)} \end{Bmatrix}$$

- Balance of heat flow
  - Node 1: $q_1^{(1)} = h_1 A(T_1^\infty - T_1)$
  - Node 2: $q_2^{(1)} + q_2^{(2)} = 0$
  - Node 3: $q_3^{(2)} = h_3 A(T_3^\infty - T_3)$
- Global matrix equation

$$\frac{kA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} h_1 A(T_1^\infty - T_1) \\ 0 \\ h_3 A(T_3^\infty - T_3) \end{Bmatrix}$$

EXAMPLE cont.

- **Move unknown nodal temperatures to LHS**

$$\begin{bmatrix} \frac{kA}{L}+h_1A & -\frac{kA}{L} & 0 \\ -\frac{kA}{L} & \frac{2kA}{L} & -\frac{kA}{L} \\ 0 & -\frac{kA}{L} & \frac{kA}{L}+h_3A \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} h_1AT_1^{\infty} \\ 0 \\ h_3AT_3^{\infty} \end{Bmatrix}$$

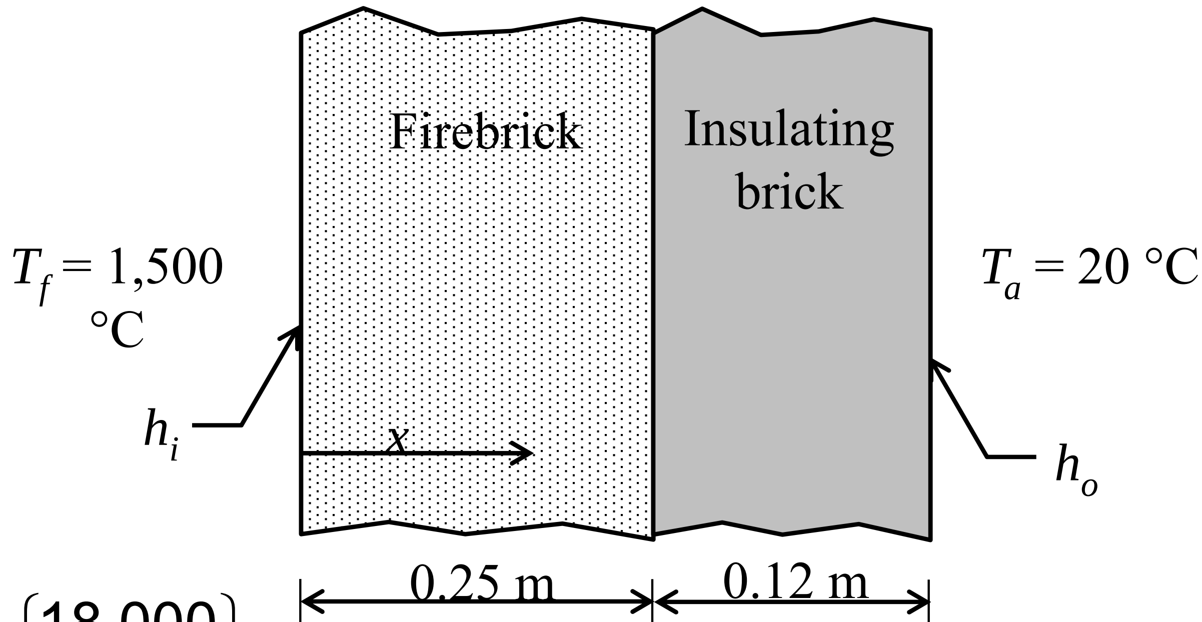
- The above matrix is P.D. because of additional positive terms in diagonal
- How much heat flow through convection boundary?
  - After solving for nodal temperature, use

$$q_1^{(1)} = h_1A(T_1^{\infty} - T_1)$$

- This is convection at the end of an element

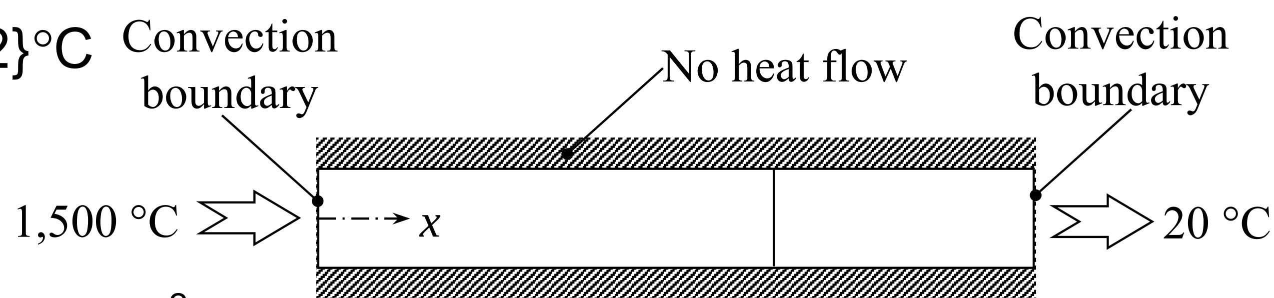
EXAMPLE: FURNACE WALL

- Firebrick
  k_1=1.2W/m/°C
  h_i=12W/m²/°C
- Insulating brick
  k_2=0.2W/m/°C
  h_o=2.0W/m²/°C

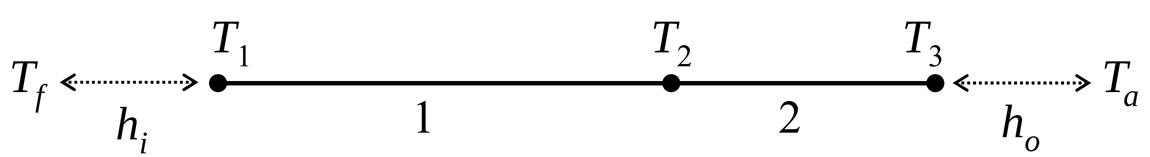


$$\begin{bmatrix} 16.8 & -4.8 & 0 \\ -4.8 & 6.47 & -1.67 \\ 0 & -1.67 & 3.67 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 18,000 \\ 0 \\ 40 \end{Bmatrix}$$

$$\{\mathbf{T}\}^T = \{1,411 \quad 1,190 \quad 552\}°C$$



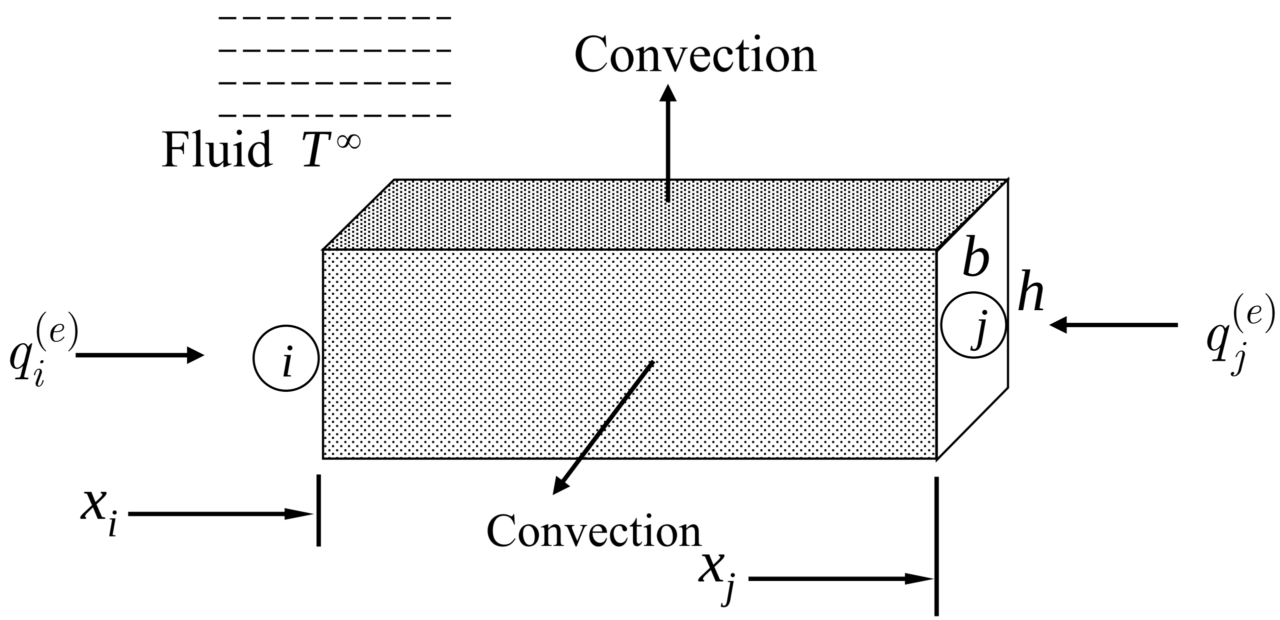
$$q_3^{(2)} = h_0(T_a - T_3) = -1054 \text{ W/m}^2$$



CONVECTION ALONG A ROD

- Long rod is submerged into a fluid
- Convection occurs across the entire surface
- Governing differential equation

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) + AQ_g + hP\left(T^{\infty} - T\right) = 0, \quad 0 \le x \le L \qquad P = 2(b+h)$$



CONVECTION ALONG A ROD cont.

- DE with approximate temperature

$$\frac{d}{dx}\left(kA\frac{d\tilde{T}}{dx}\right)+AQ_g+hP\left(T^{\infty}-\tilde{T}\right)=R(x)$$

- Minimize the residual with interpolation function $N_i(x)$

$$\int_{x_i}^{x_j}\left(\frac{d}{dx}\left(kA\frac{d\tilde{T}}{dx}\right)+AQ_g+hP(T^{\infty}-\tilde{T})\right)N_i(x)dx=0$$

- Integration by parts

$$kA\frac{d\tilde{T}}{dx}N_i(x)\bigg|_{x_i}^{x_j}-\int_{x_i}^{x_j}kA\frac{d\tilde{T}}{dx}\frac{dN_i}{dx}dx-\int_{x_i}^{x_j}hP\tilde{T}N_idx=-\int_{x_i}^{x_j}AQ_gN_i(x)dx-\int_{x_i}^{x_j}hPT^{\infty}N_idx$$

CONVECTION ALONG A ROD cont.

- Substitute interpolation scheme and rearrange

$$\int_{x_i}^{x_j} kA\left(T_i\frac{dN_i}{dx}+T_j\frac{dN_j}{dx}\right)\frac{dN_i}{dx}dx+\int_{x_i}^{x_j} hP(T_iN_i+T_jN_j)N_i\,dx$$

$$=\int_{x_i}^{x_j}(AQ_g+hPT^{\infty})N_i\,dx-q(x_j)N_i(x_j)+q(x_i)N_i(x_i)$$

- Perform integration and simplify

$$\frac{kA}{L^{(e)}}\left(T_i-T_j\right)+hpL^{(e)}\left(\frac{T_i}{3}+\frac{T_j}{6}\right)=Q_i^{(e)}+q_i^{(e)}$$

$$Q_i^{(e)}=\int_{x_i}^{x_j}(AQ_g+hPT^{\infty})N_i(x)dx$$

- Repeat the same procedure with interpolation function $N_j(x)$

CONVECTION ALONG A ROD cont.

- Finite element equation with convection along the rod

$$\left[\frac{kA}{L^{(e)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{hPL^{(e)}}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right]\begin{Bmatrix}T_i\\ T_j\end{Bmatrix}=\begin{Bmatrix}Q_i^{(e)}+q_i^{(e)}\\ Q_j^{(e)}+q_j^{(e)}\end{Bmatrix}$$

$$\left[[\mathbf{k}_T^{(e)}]+[\mathbf{k}_h^{(e)}]\right]\{\mathbf{T}\}=\{\mathbf{Q}^{(e)}\}+\{\mathbf{q}^{(e)}\}$$

- Equivalent conductance matrix due to convection

$$\left[\mathbf{k}_h^{(e)}\right]=\frac{hPL^{(e)}}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

- Thermal load vector

$$\{\mathbf{Q}^{(e)}\}=\begin{Bmatrix}Q_i\\ Q_j\end{Bmatrix}=\frac{AQ_gL^{(e)}+hPL^{(e)}T^{\infty}}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

EXAMPLE: HEAT FLOW IN A COOLING FIN

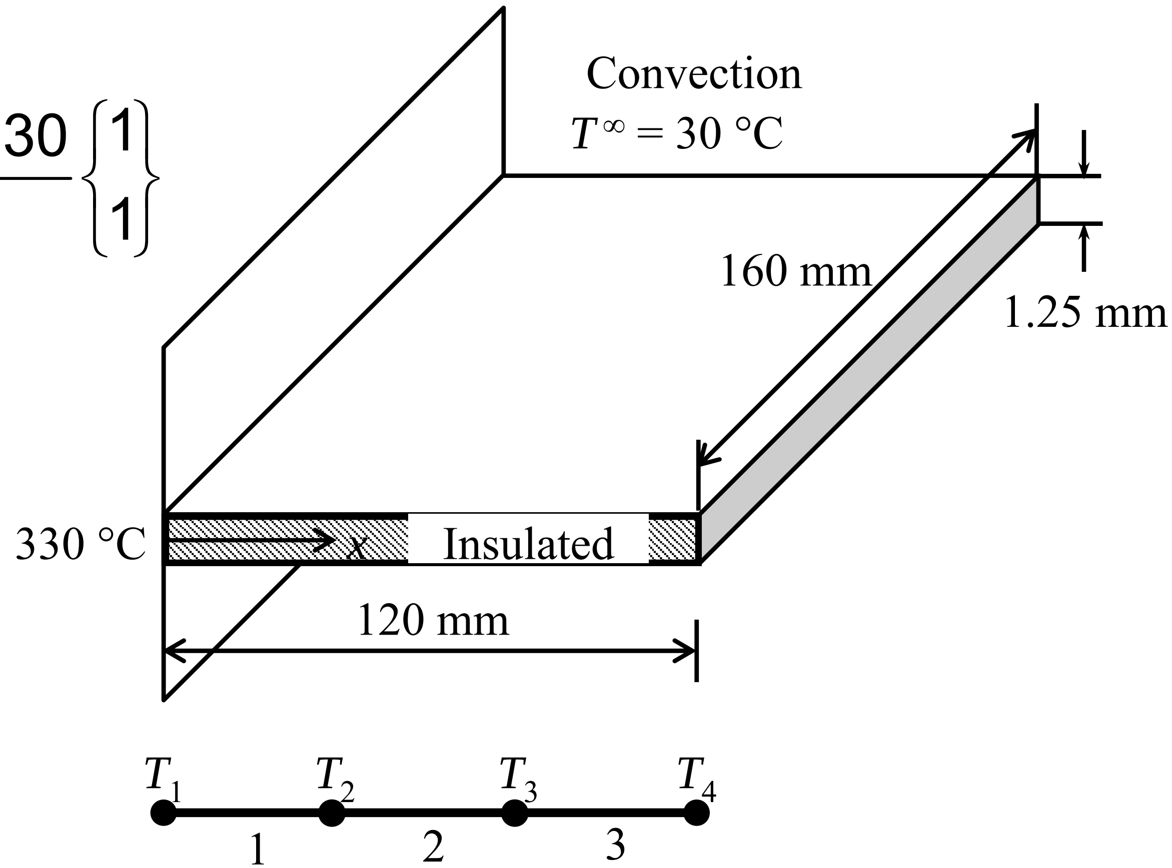
- $k = 0.2$ W/mm/°C, $h = 2\times10^{-4}$ W/mm²/°C
- Element conductance matrix

$$[\mathbf{k}_T^{(e)}] + [\mathbf{k}_h^{(e)}] = \frac{0.2\times200}{40}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{2\times10^{-4}\times320\times40}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- Thermal load vector

$$\{\mathbf{Q}^{(e)}\} = \frac{2\times10^{-4}\times320\times40\times30}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

- Element 1



EXAMPLE: HEAT FLOW IN A COOLING FIN cont.

- Element conduction equation
  - Element 1 $\begin{bmatrix} 1.8533 & -0.5733 \\ -0.5733 & 1.8533 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 38.4 \\ 38.4 \end{Bmatrix} + \begin{Bmatrix} q_1^{(1)} \\ q_2^{(1)} \end{Bmatrix}$
  - Element 2 $\begin{bmatrix} 1.8533 & -0.5733 \\ -0.5733 & 1.8533 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 38.4 \\ 38.4 \end{Bmatrix} + \begin{Bmatrix} q_2^{(2)} \\ q_3^{(2)} \end{Bmatrix}$
  - Element 3 $\begin{bmatrix} 1.8533 & -0.5733 \\ -0.5733 & 1.8533 \end{bmatrix} \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 38.4 \\ 38.4 \end{Bmatrix} + \begin{Bmatrix} q_3^{(3)} \\ q_4^{(3)} \end{Bmatrix}$
- Balance of heat flow
  - Node 1 $q_1^{(1)} = Q_1$
  - Node 2 $q_2^{(1)} + q_2^{(2)} = 0$
  - Node 3 $q_3^{(2)} + q_3^{(2)} = 0$
  - Node 4 $q_4^{(3)} = hA(T^{\infty} - T_4)$

EXAMPLE: HEAT FLOW IN A COOLING FIN cont.

- Assembly

$$\begin{bmatrix} 1.853 & -.573 & 0 & 0 \\ -.573 & 3.706 & -.573 & 0 \\ 0 & -.573 & 3.706 & -.573 \\ 0 & 0 & -.573 & 1.853 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 38.4 + Q_1 \\ 76.8 \\ 76.8 \\ 38.4 + hA(T^{\infty} - T_4) \end{Bmatrix}$$

- Move T_4 to LHS and apply known $T_1 = 330$

$$\begin{bmatrix} 1.853 & -.573 & 0 & 0 \\ -.573 & 3.706 & -.573 & 0 \\ 0 & -.573 & 3.706 & -.573 \\ 0 & 0 & -.573 & 1.893 \end{bmatrix} \begin{Bmatrix} 330 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 38.4 + Q_1 \\ 76.8 \\ 76.8 \\ 39.6 \end{Bmatrix}$$

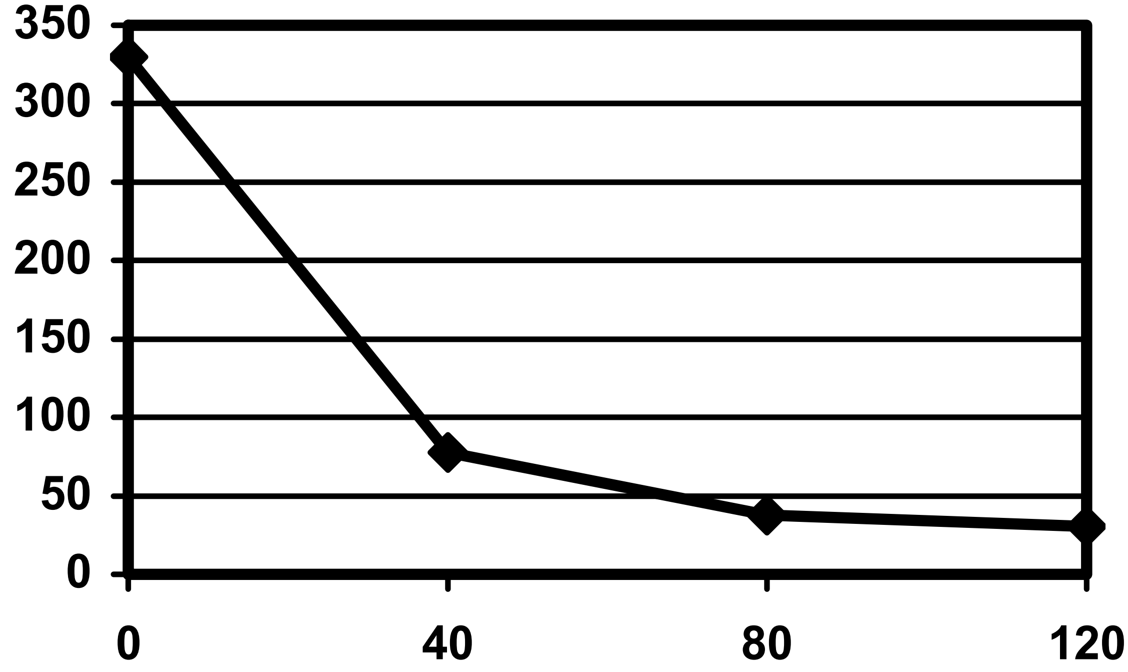
- Move the first column to RHS after multiplying with T_1=330

$$\begin{bmatrix} 3.706 & -.573 & 0 \\ -.573 & 3.706 & -.573 \\ 0 & -.573 & 1.893 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 265.89 \\ 76.8 \\ 39.6 \end{Bmatrix}$$

EXAMPLE: HEAT FLOW IN A COOLING FIN cont.

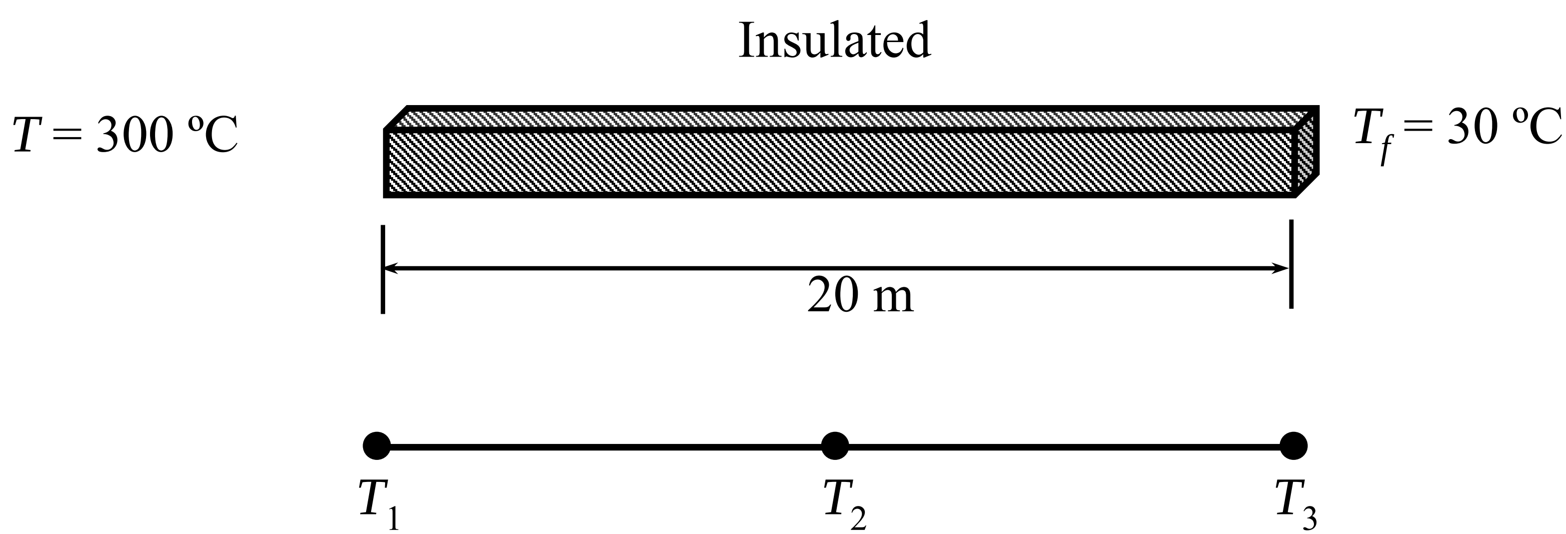
- Solve for temperature

$$T_1 = 330°C, \;\; T_2 = 77.57°C, \;\; T_3 = 37.72°C, \;\; T_4 = 32.34°C$$



Exercise

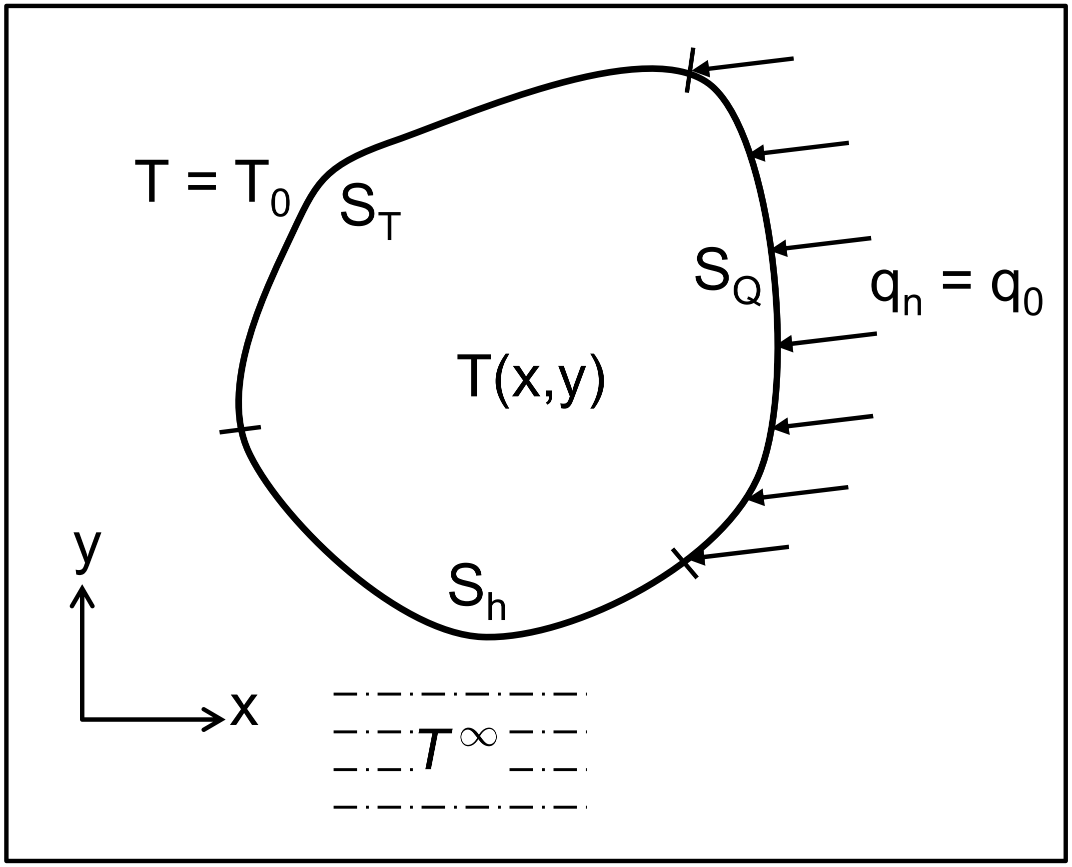
- Determine the temperature distribution (nodal temperatures) of the structure shown in the figure using two equal–length, linear finite elements with the cross-sectional area of 1 m^2. The thermal conductivity is 10 W/m/ºC. The left side is maintained at 300 ºC. The right side is subjected to heat loss by convection with h = 1 $W/m^2/^oC$ and T_f = 30 ºC. All other sides are insulated.



4.6. 2D HEAT TRANSFER

2D Heat Transfer Problem

- 2D Heat transfer: Temperature remains constant through z-coordinate
  - No heat flow in z-dir
  - S_T: prescribed temperature
  - S_Q: prescribed heat flux
  - S_h: convection boundary



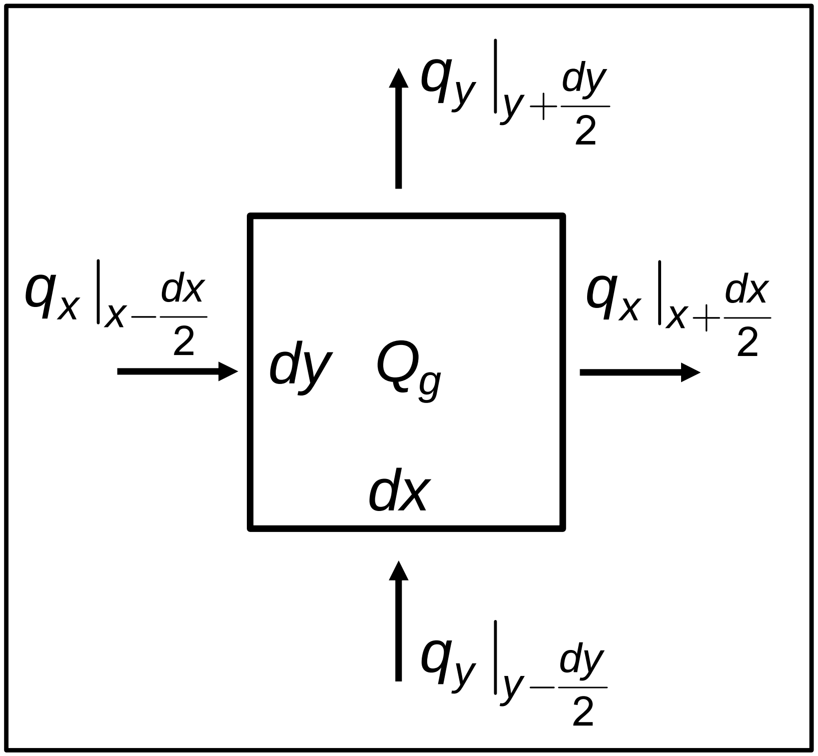
- Temperature distribution on a thin plate
- Temperature distribution over the cross-section of a nuclear reactor

Heat Balance Equation

- Conservation of energy

$E_{in} + E_{generated} = E_{out}$

$$\left(q_x\big|_{x-\frac{dx}{2}} - q_x\big|_{x+\frac{dx}{2}} \right) t_z dy + \left(q_y\big|_{y-\frac{dy}{2}} - q_y\big|_{y+\frac{dy}{2}} \right) t_z dx + Q_g t_z dx dy = 0$$



⇓

$$q_x\big|_{x-\frac{dx}{2}} - q_x\big|_{x+\frac{dx}{2}} = \left[q_x\big|_x + \frac{\partial q_x}{\partial x}\left(-\frac{dx}{2} \right) \right] - \left[q_x\big|_x + \frac{\partial q_x}{\partial x}\left(\frac{dx}{2} \right) \right] = -\frac{\partial q_x}{\partial x} dx$$

$$q_y\big|_{y-\frac{dy}{2}} - q_y\big|_{y+\frac{dy}{2}} = -\frac{\partial q_y}{\partial y} dy$$

⇒

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = Q_g$$

Constitutive Equation (Fourier's Law)

- Fourier's law: heat flux is proportional to the negative of temperature gradient

$$q_x = -k_{xx}\frac{\partial T}{\partial x} - k_{xy}\frac{\partial T}{\partial y}$$

$$q_y = -k_{xy}\frac{\partial T}{\partial x} - k_{yy}\frac{\partial T}{\partial y}$$

  - kxx, kxy, kyy: Thermal conductivities
  - isotropic material: $k_{xx} = k_{yy} = k$ and $k_{xy} = 0$

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = -[\mathbf{k}]\{\nabla T\} = -\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}\begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix}$$

Conductivity matrix

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = -k\{\nabla T\}$$

For isotropic material

We will only consider isotropic material case

Governing Differential Equation

- Fourier's law → Conservation of energy

$$\frac{\partial}{\partial x}\left(k_{xx}\frac{\partial T}{\partial x}+k_{xy}\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(k_{yx}\frac{\partial T}{\partial x}+k_{yy}\frac{\partial T}{\partial y}\right)+Q_g=0$$

$$\boxed{\nabla\cdot([\mathbf{k}]\{\nabla T\})+Q_g=0}$$

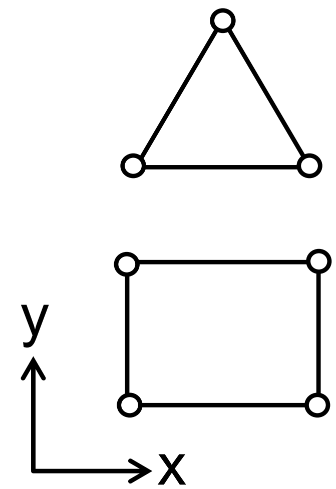
2nd-order differential equation

- Boundary conditions
  - Normal inward heat flux: $q_n=\mathbf{q}\cdot(-\mathbf{n})=(k\{\nabla T\})\cdot\mathbf{n}$
  - Temperature boundary S_T: $T=T_0$
  - Heat flux boundary S_Q: $q_n=q_0$
  - Convection boundary S_h: $q_n=h(T^{\infty}-T)$

In 2D heat transfer, we need temperature T(x, y)

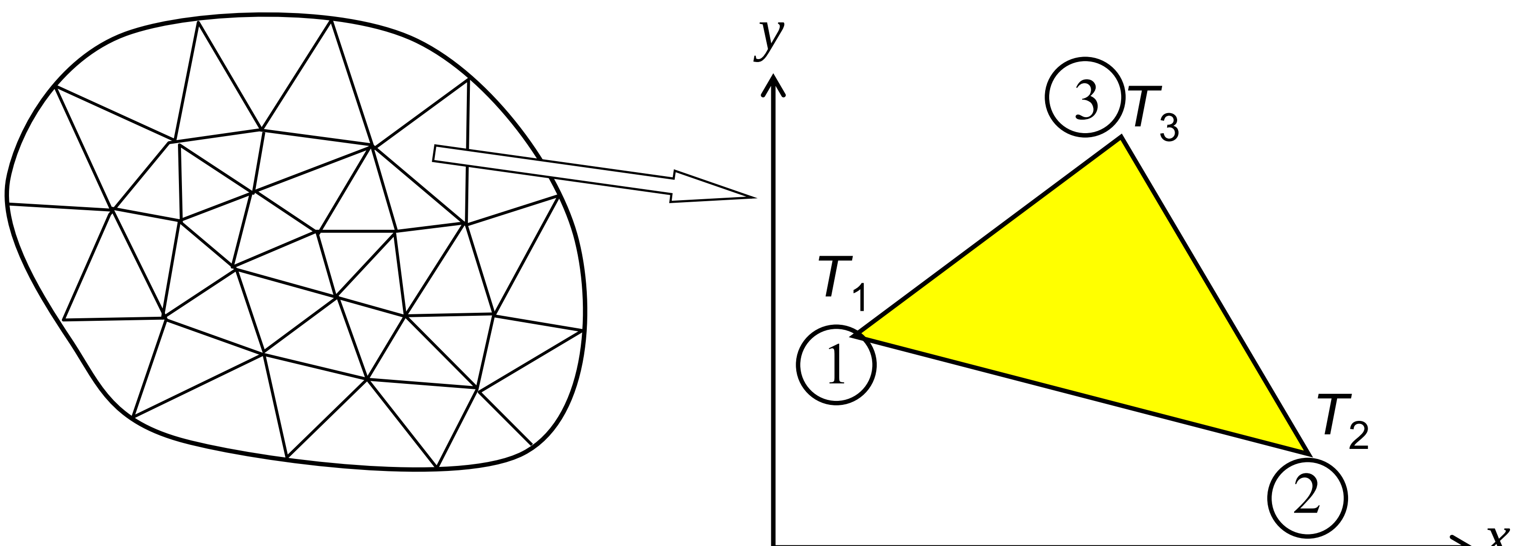
2D Finite Element Interpolation

- Let 2D domain is discretized by 2D elements
- Commonly either triangular or rectangular shapes
- Then, FE interpolation is (x, y) coordinates



$$\tilde{T}(x,y) = \sum_{k=1}^{N} N_k(x,y) T_K$$

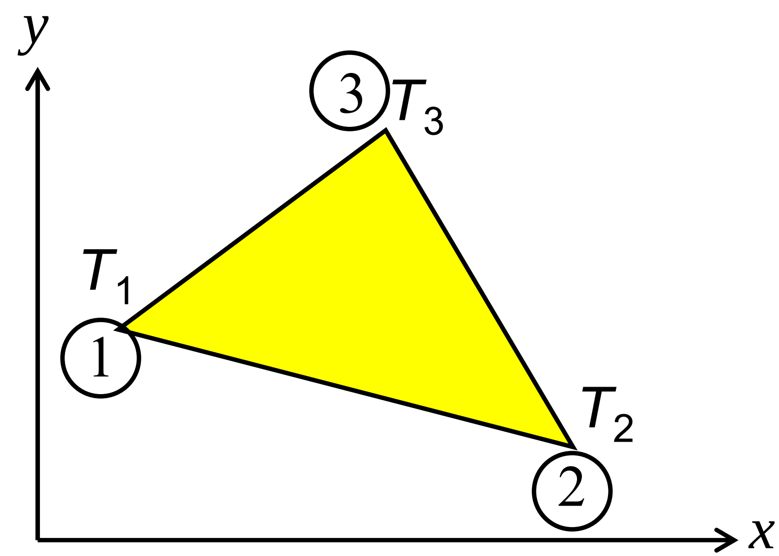
  - T_k: nodal temperature
  - $N_k(x, y)$: interpolation function



2D Triangular Element

- Interpolation is a function of *x* and *y* coordinates
- Interpolation function is a three term polynomial in *x* and *y*
  - Three nodal temperatures (T_1, T_2, T_3) are available

$$\tilde{T}(x,y) = a_1 + a_2 x + a_3 y$$



- Goal is to express in terms of nodal temperature

$$\tilde{T}(x,y) = N_1(x,y)T_1 + N_2(x,y)T_2 + N_3(x,y)T_3$$

- Interpolation requirement (property)

$$\begin{cases} \tilde{T}(x_1,y_1) \equiv T_1 = a_1 + a_2 x_1 + a_3 y_1 \\ \tilde{T}(x_2,y_2) \equiv T_2 = a_1 + a_2 x_2 + a_3 y_2 \\ \tilde{T}(x_3,y_3) \equiv T_3 = a_1 + a_2 x_3 + a_3 y_3 \end{cases}$$

Interpolation of Triangular Element

- In matrix notation

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Is the matrix singular? When?

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

$$\begin{cases} f_1 = x_2 y_3 - x_3 y_2, & b_1 = y_2 - y_3, \quad c_1 = x_3 - x_2 \\ f_2 = x_3 y_1 - x_1 y_3, & b_2 = y_3 - y_1, \quad c_2 = x_1 - x_3 \\ f_3 = x_1 y_2 - x_2 y_1, & b_3 = y_1 - y_2, \quad c_3 = x_2 - x_1 \end{cases}$$

$$A = \frac{1}{2} \det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

Area

Interpolation of Triangular Element

- Coefficients

$$a_1 = \frac{1}{2A}(f_1T_1 + f_2T_2 + f_3T_3)$$

$$a_2 = \frac{1}{2A}(b_1T_1 + b_2T_2 + b_3T_3)$$

$$a_3 = \frac{1}{2A}(c_1T_1 + c_2T_2 + c_3T_3)$$

- Interpolation equation

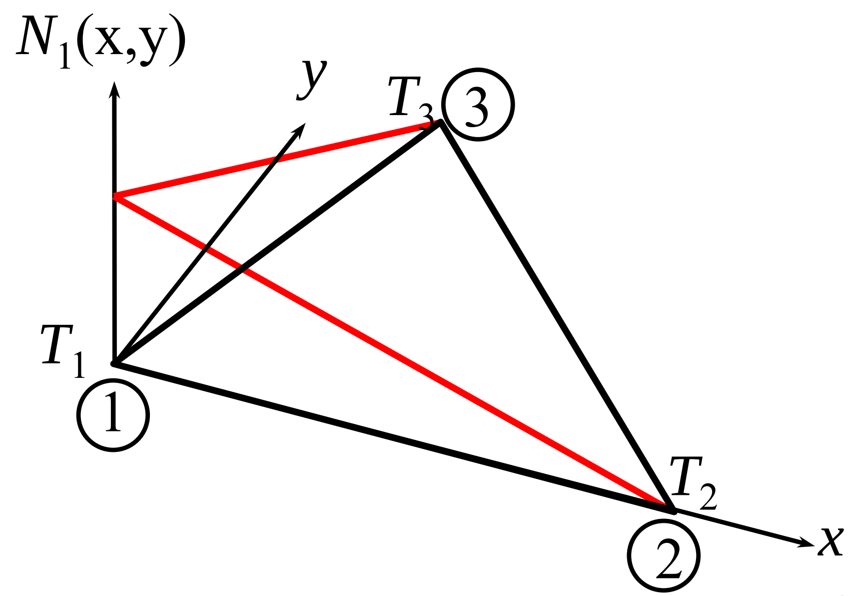
$$\tilde{T}(x,y) = a_1 + a_2x + a_3y$$

$$= \frac{1}{2A}\left[(f_1T_1 + f_2T_2 + f_3T_3) + (b_1T_1 + b_2T_2 + b_3T_3)x + (c_1T_1 + c_2T_2 + c_3T_3)y\right]$$

$$= \frac{1}{2A}(f_1 + b_1x + c_1y)T_1 \quad N_1(x,y)$$

$$+ \frac{1}{2A}(f_2 + b_2x + c_2y)T_2 \quad N_2(x,y)$$

$$+ \frac{1}{2A}(f_3 + b_3x + c_3y)T_3 \quad N_3(x,y)$$



Interpolation of Triangular Element

- Temperature interpolation of triangular element

$$\tilde{T}(x,y) = \sum_{k=1}^{3} N_k(x,y) T_k = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \lfloor \mathbf{N} \rfloor \{\mathbf{T}\}$$

$$\begin{cases} N_1(x,y) = \dfrac{1}{2A}(f_1 + b_1 x + c_1 y) \\ N_2(x,y) = \dfrac{1}{2A}(f_2 + b_2 x + c_2 y) \\ N_3(x,y) = \dfrac{1}{2A}(f_3 + b_3 x + c_3 y) \end{cases}$$

Shape Function

- N_1, N_2, and N_3 are linear functions of x- and y-coordinates.
- Interpolated temperature changes linearly along the each coordinate direction.

Interpolation and Heat Flux

- Derivatives of temperature

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left(\sum_{k=1}^{3} N_k(x,y)T_k\right) = \sum_{k=1}^{3}\frac{\partial N_k}{\partial x}T_k = \sum_{k=1}^{3}\frac{b_k}{2A}T_k$$

$$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(\sum_{k=1}^{3} N_k(x,y)T_k\right) = \sum_{k=1}^{3}\frac{\partial N_k}{\partial y}T_k = \sum_{k=1}^{3}\frac{c_k}{2A}T_k$$

$$\Rightarrow \{\nabla T\} = \begin{Bmatrix} \partial T / \partial x \\ \partial T / \partial y \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_2 \end{Bmatrix} = [\mathbf{B}]\{\mathbf{T}\}$$

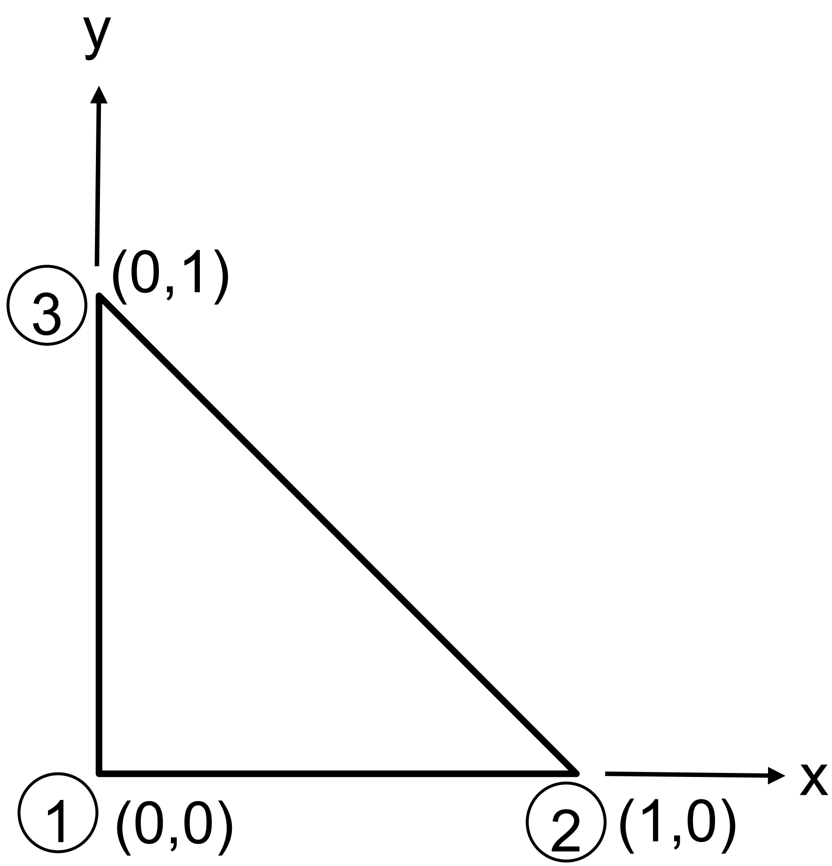
- Heat flux

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = -k\{\nabla T\} = -\mathrm{k}[\mathbf{B}]\{\mathbf{T}\}$$

  – Heat flux is constant within an element

Exercise

- Build shape functions of the triangular element shown in the figure and express functional expression of temperature and heat flux (q_x, q_y). Use T_1 = 50, T_2 = 80, T_3 = 30 ^{o}C, t = 1 m, k = 10 W/m/^{o}C.



Galerkin Method for Heat Conduction

- 2D weighted residual form:

$$\iint_A R(x,y)W(x,y)t\,\mathrm{d}A = 0$$

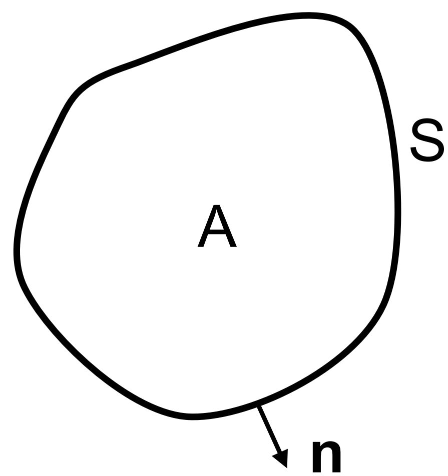
$$\Rightarrow \iint_A \left[k\left(\frac{\partial^2\tilde{T}}{\partial x^2}+\frac{\partial^2\tilde{T}}{\partial y^2}\right)+Q_g\right]\phi_k t\,\mathrm{d}A = 0$$

ϕ_k: trial function
$k = 1,\ldots,N$

$$\iint_A \left(k\nabla^2\tilde{T}+Q_g\right)\phi_k t\,\mathrm{d}A = 0$$

- Green's theorem (integration by parts)

$$\iint_A (\nabla^2 u)v\,\mathrm{d}A = \int_S (\nabla u\cdot\mathbf{n})v\,\mathrm{d}S - \iint_A (\nabla u\cdot\nabla v)\mathrm{d}A$$



unit normal vector

Galerkin Method for Heat Conduction

- Apply Green's theorem to the Galerkin's method

$$\iint_A \left(k\nabla^2\tilde{T} + Q_g\right)\phi_k t\,\mathrm{d}A = 0$$

$$\Rightarrow \int_S k(\nabla T\cdot\mathbf{n})\phi_k t\,\mathrm{d}S - \iint_A \left(k\nabla\tilde{T}\cdot\nabla\phi_k\right)t\,\mathrm{d}A + \iint_A Q_g\phi_k t\,\mathrm{d}A = 0$$

  - Apply boundary heat flux and rearrange

$$\Rightarrow \iint_A \left(k\nabla\tilde{T}\cdot\nabla\phi_k\right)t\,\mathrm{d}A = \iint_A Q_g\phi_k t\,\mathrm{d}A + \int_S q_n\phi_k t\,\mathrm{d}S$$

  - Divide the boundary into $S = S_T \cup S_Q \cup S_h$

$$\int_S q_n\phi_k t\,\mathrm{d}S = \int_{S_T} q_n\phi_k t\,\mathrm{d}S + \int_{S_Q} q_n\phi_k t\,\mathrm{d}S + \int_{S_h} q_n\phi_k t\,\mathrm{d}S$$

$$= \int_{S_Q} q_0\phi_k t\,\mathrm{d}S + \int_{S_h} h(T^\infty - T)\phi_k t\,\mathrm{d}S$$

$$\Rightarrow \iint_A \left(k\nabla\tilde{T}\cdot\nabla\phi_k\right)t\,\mathrm{d}A = \iint_A Q_g\phi_k t\,\mathrm{d}A + \int_{S_Q} q_0\phi_k t\,\mathrm{d}S + \int_{S_h} h(T^\infty - T)\phi_k t\,\mathrm{d}S$$

Finite Element Formulation

- Consider triangular element $A^{(e)}$ and shape functions $N_k(x, y)$
- LHS (conductivity part) with k = 1, 2, 3

$$\iint_{A^{(e)}} \left(k \begin{bmatrix} \nabla N_1 \\ \nabla N_2 \\ \nabla N_3 \end{bmatrix} \cdot \nabla \tilde{T} \right) t \, \mathrm{d}A = \underbrace{\left[\iint_{A^{(e)}} \underbrace{\left(k[\mathbf{B}]^{\mathrm{T}}[\mathbf{B}] \right) t}_{\text{constant}} \, \mathrm{d}A \right]}_{[\mathbf{k}^{(e)}] \text{ element stiffness matrix}} \{\mathbf{T}\}$$

$$\begin{aligned}
[\mathbf{k}^{(e)}] &= ktA[\mathbf{B}]^{\mathrm{T}}[\mathbf{B}] \\
&= ktA \frac{1}{2A} \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} \cdot \frac{1}{2A} \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \\
&= \frac{kt}{4A} \begin{bmatrix} b_1^2 + c_1^2 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_1 b_2 + c_1 c_2 & b_2^2 + c_2^2 & b_2 b_3 + c_2 c_3 \\ b_1 b_3 + c_1 c_3 & b_2 b_3 + c_2 c_3 & b_3^2 + c_3^2 \end{bmatrix}
\end{aligned}$$

Finite Element Formulation

- Load due to distributed heat source: k = 1, 2, 3
  - Ex) heat generated due to electrical current flow

$$\iint_{A^{(e)}} Q_g \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} t\,\mathrm{d}A = \begin{Bmatrix} Q_{g1} \\ Q_{g2} \\ Q_{g3} \end{Bmatrix} = \{\mathbf{Q}_g^{(e)}\}$$

- Uniform heat generation: Q_g = constant

$$\int_{A^{(e)}} N_k \,\mathrm{d}A = \frac{1}{3} A^{(e)}$$

$$\{\mathbf{Q}_g^{(e)}\} = \iint_{A^{(e)}} Q_g \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} t\,\mathrm{d}A = \frac{1}{3} Q_g A^{(e)} t \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

- This nodal heat sources should be assembled to the global heat source vector

Finite Element Formulation

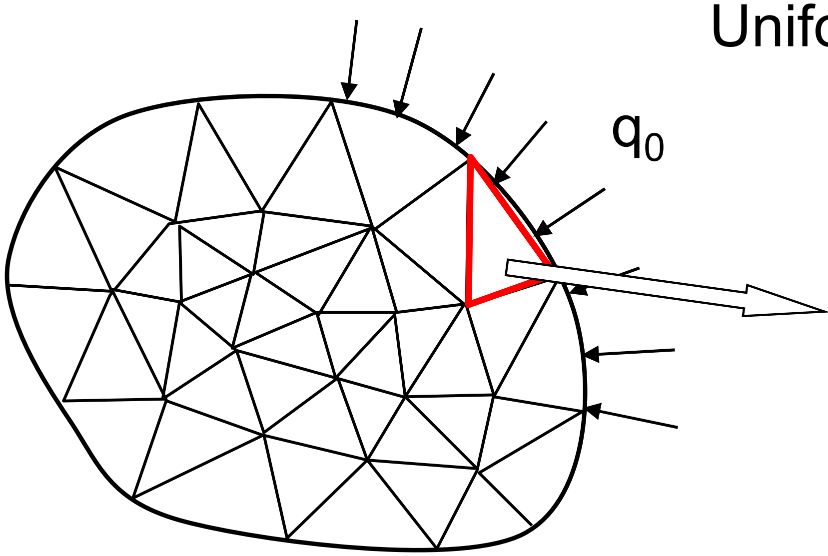
- Load due to distributed heat flux on S_Q
  - If element (e) is within a domain, there is no S_Q boundary
  - Only elements on the boundary may have a part of S_Q boundary
  - Let's assume edge 1-2 of element (e) is on the heat flux boundary
  - N_3 = 0 along edge 1-2

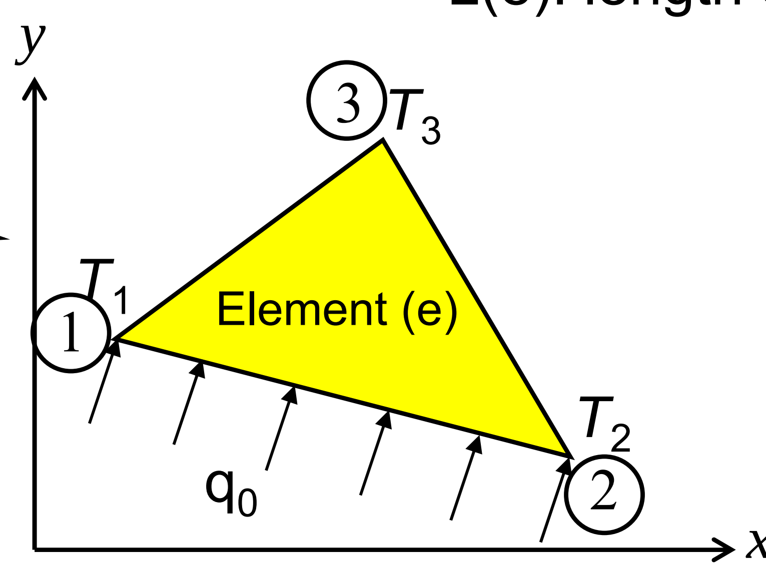
$$\{\mathbf{Q}_Q^{(e)}\} = \int_{S_Q^{(e)}} q_0 \begin{Bmatrix} N_1 \\ N_2 \\ 0 \end{Bmatrix} t \, dS \qquad \{\mathbf{Q}_Q^{(e)}\} = \frac{q_0 t L^{(e)}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \qquad \int_{S^{(e)}} N_k \, dS = \frac{1}{2} L^{(e)}$$

Uniform heat flux

k = 1, 2

L(e): length of edge 1-2





Finite Element Formulation

- Convection boundary condition
  - Let's assume edge 1-2 of element (e) is on the convection boundary
  - $N_3 = 0$ along edge 1-2
  - Same as 1D convection, stiffness and heat load terms

$$\{\mathbf{Q}_h^{(e)}\} = htT^{\infty}\int_{S_h}\begin{Bmatrix} N_1 \\ N_2 \\ 0 \end{Bmatrix} \mathrm{d}S = \frac{htL^{(e)}T^{\infty}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

$$ht\int_{S_h^{(e)}} T\begin{Bmatrix} N_1 \\ N_2 \\ 0 \end{Bmatrix} \mathrm{d}S = ht\int_{S_h^{(e)}}\begin{Bmatrix} N_1 \\ N_2 \\ 0 \end{Bmatrix}\lfloor N_1 \quad N_2 \quad 0 \rfloor \mathrm{d}S\{\mathbf{T}\}$$

$$[\mathbf{k}_T^{(e)}] = \frac{htL^{(e)}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Add to the heat conduction matrix

Finite Element Equation for Heat Transfer

- Element equation
  - With constant heat source, heat flux on Edge 1-2, and convection on Edge 1-2

$$\left[\frac{kt}{4A}\begin{bmatrix} b_1^2+c_1^2 & b_1b_2+c_1c_2 & b_1b_3+c_1c_3 \\ b_1b_2+c_1c_2 & b_2^2+c_2^2 & b_2b_3+c_2c_3 \\ b_1b_3+c_1c_3 & b_2b_3+c_2c_3 & b_3^2+c_3^2 \end{bmatrix} + \frac{htL^{(e)}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

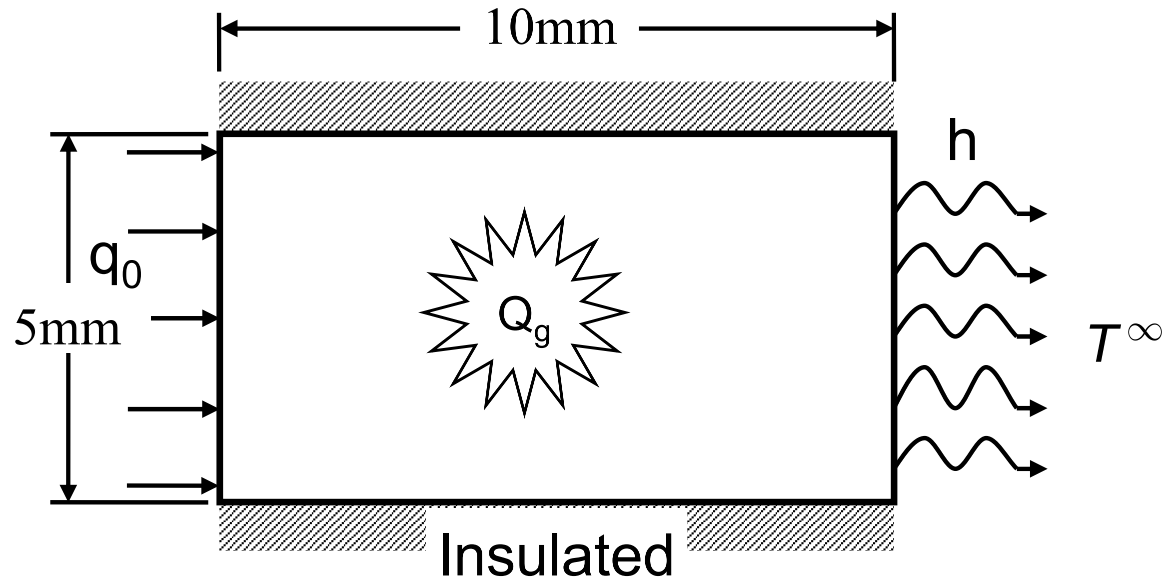
$$= \frac{1}{3}Q_g A^{(e)} t \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} + \frac{q_0 t L^{(e)}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} + \frac{htL^{(e)}T^{\infty}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

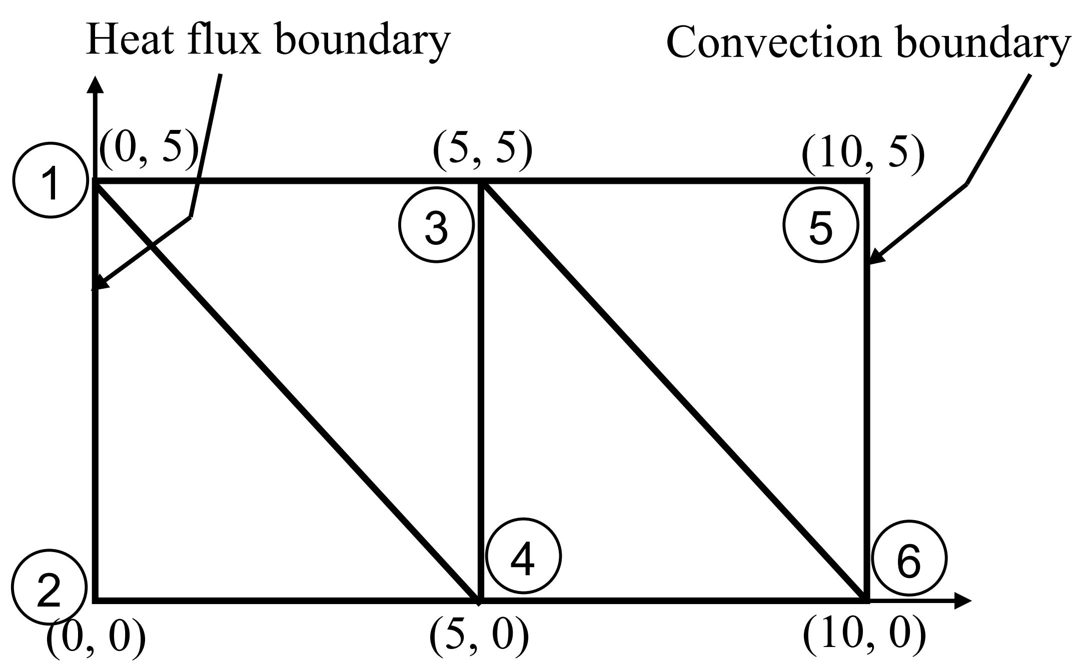
Ex) Heat Transfer along a Conducting Block

- Determine temperature distribution using triangular elements

- Thickness = 5 mm, $k = 0.2$ W/mm/^oC, $Q_g = 0.06$ W/mm^3

- $q_0 = 0.04$ W/mm^2, $h = 0.012$ W/mm^2/^oC, $T^\infty = 25^oC$





Ex) Heat Transfer along a Conducting Block

- Element connectivity table

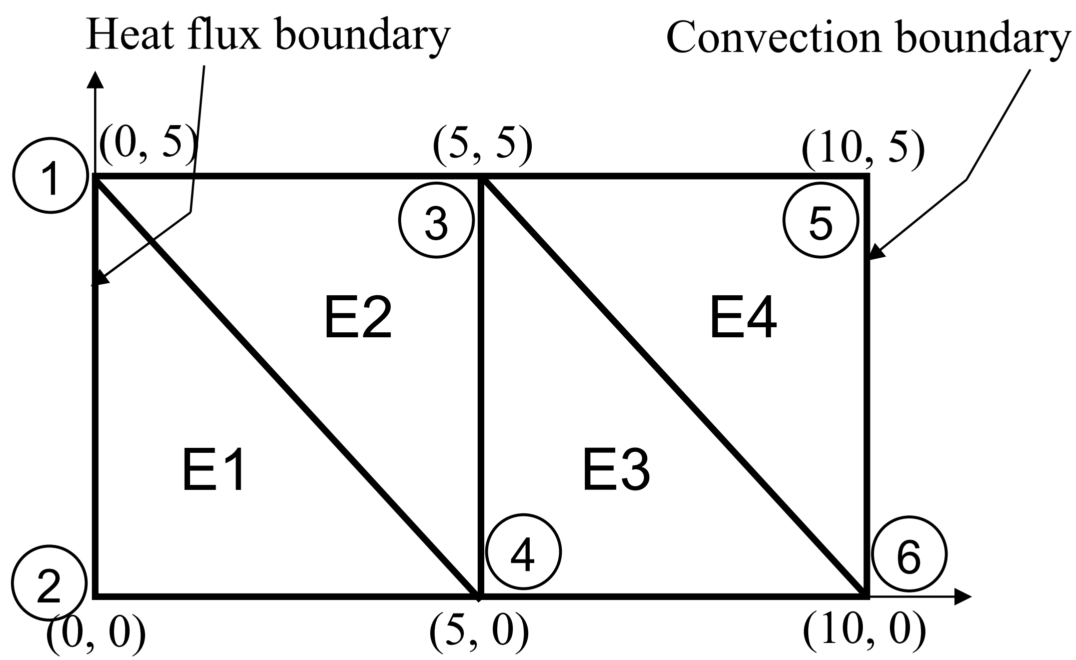
Element	LN1	LN2	LN3
1	1	2	4
2	1	4	3
3	3	4	6
4	3	6	5

- Element 1

$$b_1 = y_2 - y_3 = 0, \quad c_1 = x_3 - x_2 = 5$$
$$b_2 = y_3 - y_1 = -5, \quad c_2 = x_1 - x_3 = -5$$
$$b_3 = y_1 - y_2 = 5, \quad c_3 = x_2 - x_1 = 0$$

$$A^{(e)} = \frac{1}{2}\det\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{25}{2}$$

$$[\mathbf{B}] = \frac{1}{2A^{(e)}}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 5 & -5 & 0 \end{bmatrix}$$



Ex) Heat Transfer along a Conducting Block

- Element stiffness matrix (E1 and E3)

$$[\mathbf{K}_1] = [\mathbf{K}_3] = ktA^{(e)}[\mathbf{B}]^T[\mathbf{B}] = 0.5\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

- Element stiffness matrix (E2 and E4)

$$[\mathbf{K}_2] = [\mathbf{K}_4] = ktA^{(e)}[\mathbf{B}]^T[\mathbf{B}] = 0.5\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

- Heat generation (for all elements)

$$\{\mathbf{Q}_g^{(e)}\} = \frac{1}{3}Q_g A^{(e)} t \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{1}{3}\cdot(0.06)\cdot\left(\frac{25}{2}\right)\cdot(5)\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 1.25\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Ex) Heat Transfer along a Conducting Block

- Heat flux (along Edge 1-2 of Element 1)

$$\{\mathbf{Q}_Q^{(1)}\} = \frac{q_0 t L^{(1)}}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix} = \frac{0.04\cdot 5\cdot 5}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix} = 0.5\begin{Bmatrix}1\\1\\0\end{Bmatrix}$$

- Convection (along Edge 5-6 of Element 4)

$$\{\mathbf{Q}_h^{(4)}\} = \frac{htL^{(4)}T^{\infty}}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix} = \frac{0.012\cdot 5\cdot 5\cdot 25}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix} = \begin{Bmatrix}0\\3.75\\3.75\end{Bmatrix}$$

$$[\mathbf{k}_T^{(e)}] = \frac{htL^{(4)}}{6}\begin{bmatrix}0 & 0 & 0\\0 & 2 & 1\\0 & 1 & 2\end{bmatrix} = \frac{0.012\cdot 5\cdot 5}{6}\begin{bmatrix}0 & 0 & 0\\0 & 2 & 1\\0 & 1 & 2\end{bmatrix} = 0.05\begin{bmatrix}0 & 0 & 0\\0 & 2 & 1\\0 & 1 & 2\end{bmatrix}$$

Ex) Heat Transfer along a Conducting Block

- Assembly

$$0.5\begin{bmatrix} 1+1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2+1+1 & -1-1 & -1 & 0 \\ 0 & -1 & -1-1 & 1+2+1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2+0.2 & -1+0.1 \\ 0 & 0 & 0 & -1 & -1+0.1 & 1+1+0.2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = 1.25\begin{Bmatrix} 1+1+0.4 \\ 1+0.4 \\ 1+1+1 \\ 1+1+1 \\ 1+3 \\ 1+1+3 \end{Bmatrix}$$

- E1: E2: E3: E4:
- Solution: (no essential BC)

$$\{T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6\} = \{95.75 \quad 94.92 \quad 90.59 \quad 90.58 \quad 77.93 \quad 78.75\}$$

- Heat flux Edge 1-2 (constant for Element 1)

$$\mathbf{q} = \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = \frac{-k}{2A_e}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_4 \end{Bmatrix} = \frac{-0.2}{25}\begin{bmatrix} 0 & -5 & 5 \\ 5 & -5 & 0 \end{bmatrix}\begin{Bmatrix} 95.75 \\ 94.92 \\ 90.58 \end{Bmatrix} = \begin{Bmatrix} 0.174 \\ -0.0332 \end{Bmatrix}$$

Exercise

- When temperature T_1 = 20^{o}C and heat generation Q_g = 12 W/m^3, calculate the temperature T_2 and T_3 as well as heat flux at Node 1. Use k = 0.1 W/m/^{o}C and t = 0.1 m.