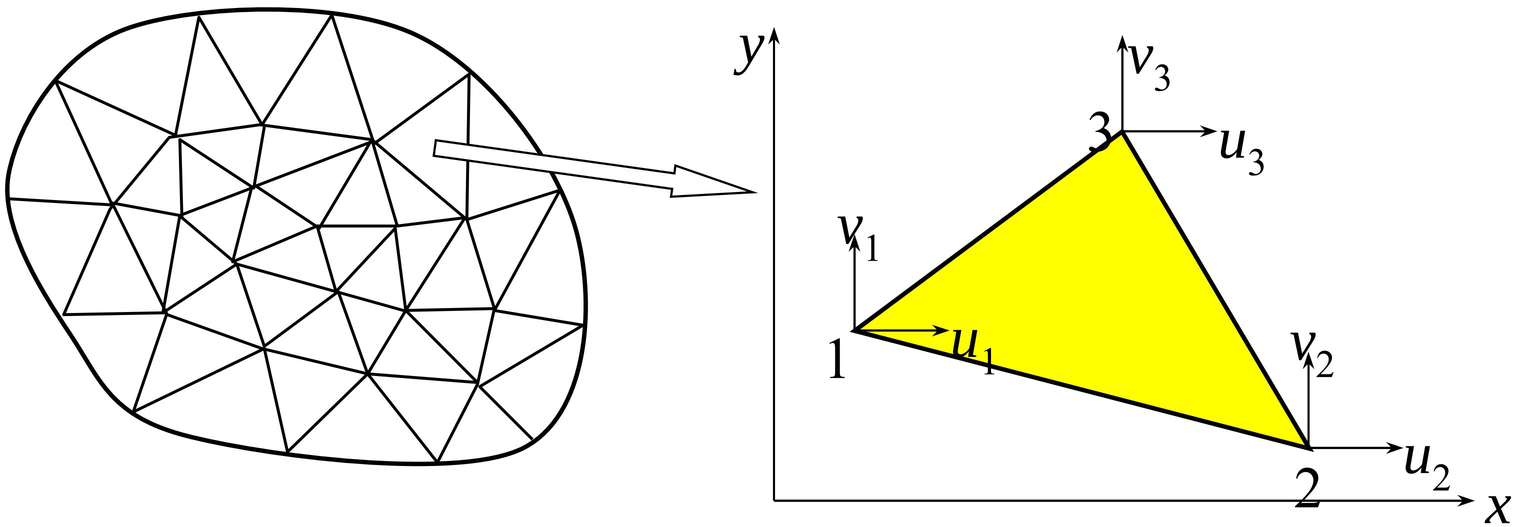
CHAP 6 FINITE ELEMENTS FOR PLANE SOLIDS

CST ELEMENT

- **Constant Strain Triangular** Element
  - Decompose two-dimensional domain by a set of triangles.
  - Each triangular element is composed by **three corner nodes**.
  - Each element shares its edge and two corner nodes with an adjacent element
  - Counter-clockwise or clockwise node numbering
  - **Each node has two DOFs: *u* and *v***
  - displacements interpolation using the shape functions and nodal displacements.
  - Displacement is linear because three nodal data are available.
  - Stress & strain are constant.



CST ELEMENT *cont.*

- Displacement Interpolation
  - Since two-coordinates are perpendicular, $u(x,y)$ and $v(x,y)$ are separated.
  - $u(x,y)$ needs to be interpolated in terms of u_1, u_2, and u_3, and $v(x,y)$ in terms of v_1, v_2, and v_3.
  - interpolation function must be a three term polynomial in x and y.
  - Since we must have rigid body displacements and constant strain terms in the interpolation function, the displacement interpolation must be of the form

$$\begin{cases} u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y \\ v(x,y) = \beta_1 + \beta_2 x + \beta_3 y \end{cases}$$

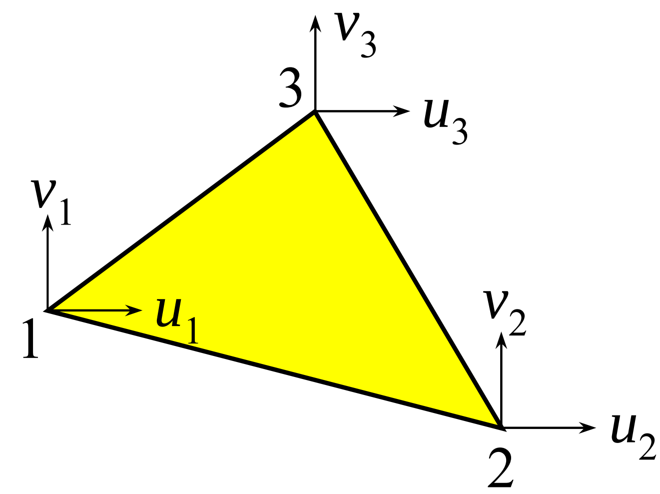
  - The goal is how to calculate unknown coefficients α_i and β_i, i = 1, 2, 3, in terms of nodal displacements.

$$u(x,y) = N_1(x,y)u_1 + N_2(x,y)u_2 + N_3(x,y)u_3$$

CST ELEMENT *cont.*

- Displacement Interpolation
  - *x*-displacement: Evaluate displacement at each node

$$\begin{cases} u(x_1,y_1) \equiv u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 \\ u(x_2,y_2) \equiv u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 \\ u(x_3,y_3) \equiv u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 \end{cases}$$



  - In matrix notation

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

  - Is the coefficient matrix singular?

CST ELEMENT *cont.*

- Displacement Interpolation

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

  – where

$$\begin{cases} f_1 = x_2 y_3 - x_3 y_2, & b_1 = y_2 - y_3, & c_1 = x_3 - x_2 \\ f_2 = x_3 y_1 - x_1 y_3, & b_2 = y_3 - y_1, & c_2 = x_1 - x_3 \\ f_3 = x_1 y_2 - x_2 y_1, & b_3 = y_1 - y_2, & c_3 = x_2 - x_1 \end{cases}$$

  – Area:

$$A = \frac{1}{2} \det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

CST ELEMENT *cont.*

$$\alpha_1 = \frac{1}{2A}(f_1 u_1 + f_2 u_2 + f_3 u_3)$$

$$\alpha_2 = \frac{1}{2A}(b_1 u_1 + b_2 u_2 + b_3 u_3)$$

$$\alpha_3 = \frac{1}{2A}(c_1 u_1 + c_2 u_2 + c_3 u_3)$$

- Insert to the interpolation equation

$$\begin{aligned} u(x,y) &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ &= \frac{1}{2A}\left[(f_1 u_1 + f_2 u_2 + f_3 u_3) + (b_1 u_1 + b_2 u_2 + b_3 u_3)x + (c_1 u_1 + c_2 u_2 + c_3 u_3)y\right] \\ &= \boxed{\frac{1}{2A}(f_1 + b_1 x + c_1 y)} u_1 \\ &\quad + \boxed{\frac{1}{2A}(f_2 + b_2 x + c_2 y)} u_2 \\ &\quad + \boxed{\frac{1}{2A}(f_3 + b_3 x + c_3 y)} u_3 \end{aligned}$$

$N_1(x,y)$

$N_2(x,y)$

$N_3(x,y)$

CST ELEMENT *cont.*

- Displacement Interpolation
  - A similar procedure can be applied for *y*-displacement $v(x, y)$.

$$u(x,y) = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$v(x,y) = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\begin{cases} N_1(x,y) = \dfrac{1}{2A}(f_1 + b_1 x + c_1 y) \\ N_2(x,y) = \dfrac{1}{2A}(f_2 + b_2 x + c_2 y) \\ N_3(x,y) = \dfrac{1}{2A}(f_3 + b_3 x + c_3 y) \end{cases}$$

Shape Function

  - N_1, N_2, and N_3 are linear functions of *x*- and *y*-coordinates.
  - Interpolated displacement changes linearly along the each coordinate direction.

CST ELEMENT *cont.*

- Displacement Interpolation
  - Matrix Notation

$$\{\mathbf{u}\} \equiv \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\boxed{\{\mathbf{u}(x,y)\} = [\mathbf{N}(x,y)]\{\mathbf{q}\}}$$

  - [**N**]: 2×6 matrix, {**q**}: 6×1 vector.
  - For a given point (x,y) within element, calculate [N] and multiply it with {q} to evaluate displacement at the point (x,y).

CST ELEMENT *cont.*

- Strain Interpolation
  - differentiating the displacement in x- and y-directions.
  - differentiating shape function [**N**] because {**q**} is constant.

$$\varepsilon_{xx} \equiv \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} N_i(x,y)u_i\right) = \sum_{i=1}^{3}\frac{\partial N_i}{\partial x}u_i = \sum_{i=1}^{3}\frac{b_i}{2A}u_i$$

$$\varepsilon_{yy} \equiv \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\sum_{i=1}^{3} N_i(x,y)v_i\right) = \sum_{i=1}^{3}\frac{\partial N_i}{\partial y}v_i = \sum_{i=1}^{3}\frac{c_i}{2A}v_i$$

$$\gamma_{xy} \equiv \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \sum_{i=1}^{3}\frac{c_i}{2A}u_i + \sum_{i=1}^{3}\frac{b_i}{2A}v_i$$

$$\begin{cases} N_1(x,y) = \dfrac{1}{2A}(f_1 + b_1x + c_1y) \\ N_2(x,y) = \dfrac{1}{2A}(f_2 + b_2x + c_2y) \\ N_3(x,y) = \dfrac{1}{2A}(f_3 + b_3x + c_3y) \end{cases}$$

CST ELEMENT *cont.*

- Strain Interpolation

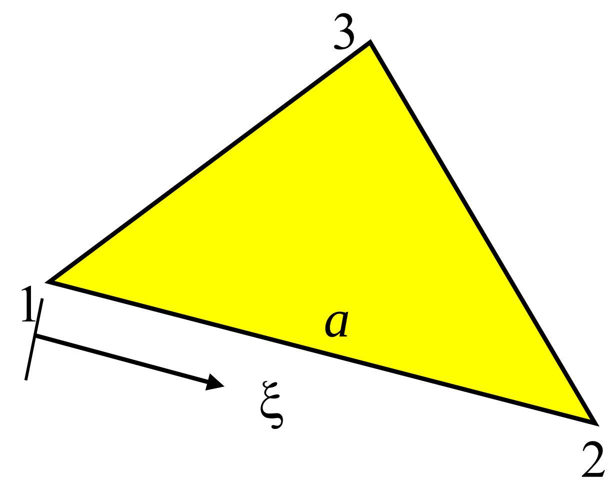
$$\{\varepsilon\} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \equiv [\mathbf{B}]\{\mathbf{q}\}$$

  – [**B**] matrix is a constant matrix and depends only on the coordinates of the three nodes of the triangular element.
  – the strains will be constant over a given element

CST ELEMENT *cont.*

- Property of CST Element
  - Since displacement is linear in *x* and *y*, the triangular element deforms into another triangle when forces are applied.
  - an imaginary straight line drawn within an element before deformation becomes another straight line after deformation.
  - Consider a local coordinate ξ such that ξ = 0 at Node 1 and ξ = *a* at Node 2.
  - Displacement on the edge 1-2:

$$\begin{cases} u(\xi) = \gamma_1 + \gamma_2 \xi \\ v(\xi) = \gamma_3 + \gamma_4 \xi \end{cases}$$

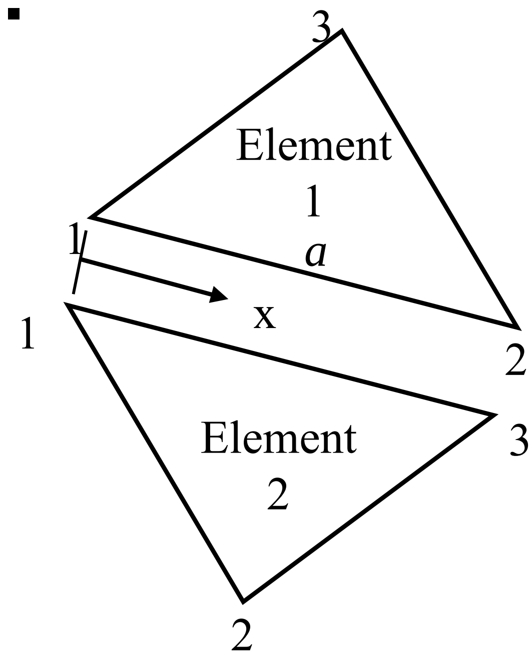


  - Since the variation of displacement is linear, the displacements should depend only on u_1 and u_2, and not on u_3.

CST ELEMENT *cont.*

- Property of CST Element

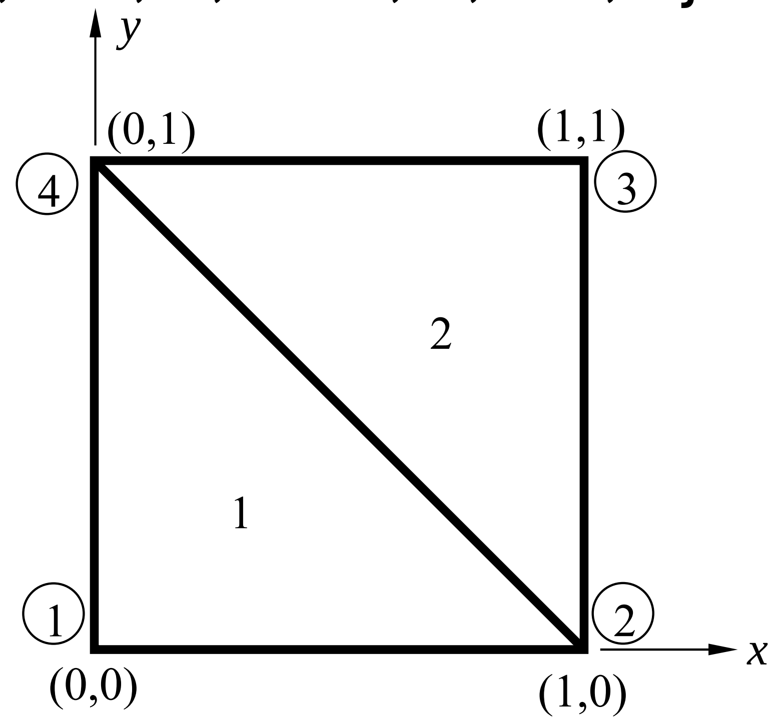
$$\begin{cases} u(\xi) = \left(1 - \dfrac{\xi}{a}\right) u_1 + \dfrac{\xi}{a} u_2 = H_1(\xi) u_1 + H_2(\xi) u_2 \\ v(\xi) = \left(1 - \dfrac{\xi}{a}\right) v_1 + \dfrac{\xi}{a} v_2 = H_1(\xi) v_1 + H_2(\xi) v_2 \end{cases}$$



- Inter-element Displacement Compatibility
  - Displacements at any point in an element can be computed from nodal displacements of that particular element and the shape functions.
  - Consider a point on a common edge of two adjacent elements, which can be considered as belonging to either of the elements.
  - Then the nodes of either triangle can be used in interpolating the displacements of this point.
  - However, one must obtain a unique set of displacements independent of the choice of the element.
  - This can be true only if the displacements of the points depend only on the nodes common to both elements.

EXAMPLE - Interpolation

- nodal displacements
  $\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{-0.1, 0, 0.1, 0, -0.1, 0, 0.1, 0\}$
- Element 1: Nodes 1-2-4



$$
\begin{array}{lll}
x_1 = 0 & x_2 = 1 & x_3 = 0 \\
y_1 = 0 & y_2 = 0 & y_3 = 1 \\
f_1 = 1 & f_2 = 0 & f_3 = 0 \\
b_1 = -1 & b_2 = 1 & b_3 = 0 \\
c_1 = -1 & c_2 = 0 & c_3 = 1
\end{array}
$$

$$N_1(x,y) = 1 - x - y$$

$$N_2(x,y) = x$$

$$N_3(x,y) = y$$

$$u^{(1)}(x,y) = \sum_{I=1}^{3} N_I(x,y)u_I = 0.1(2x + 2y - 1)$$

$$v^{(1)}(x,y) = \sum_{I=1}^{3} N_I(x,y)v_I = 0.0$$

$\Rightarrow$

$$\varepsilon_{xx}^{(1)} = \frac{\partial u^{(1)}}{\partial x} = 0.2$$

$$\varepsilon_{yy}^{(1)} = \frac{\partial v^{(1)}}{\partial y} = 0.0$$

$$\gamma_{xy}^{(1)} = \frac{\partial u^{(1)}}{\partial y} + \frac{\partial v^{(1)}}{\partial x} = 0.2$$

EXAMPLE – Interpolation cont.

- Element 2: Nodes 2-3-4

$x_1 = 1 \quad x_2 = 1 \quad x_3 = 0$

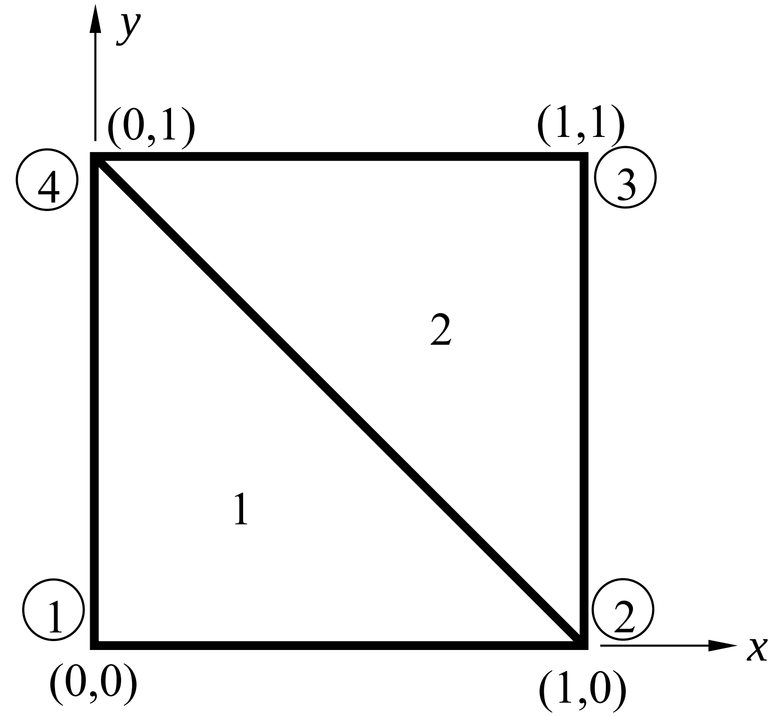
$y_1 = 0 \quad y_2 = 1 \quad y_3 = 1$

$f_1 = 1 \quad f_2 = -1 \quad f_3 = 1$

$b_1 = 0 \quad b_2 = 1 \quad b_3 = -1$

$c_1 = -1 \quad c_2 = 1 \quad c_3 = 0$

$$N_1(x,y) = 1 - y$$
$$N_2(x,y) = x + y - 1$$
$$N_3(x,y) = 1 - x$$



$$u^{(2)}(x,y) = \sum_{i=1}^{3} N_i(x,y) u_i = 0.1(3 - 2x - 2y)$$

$$v^{(2)}(x,y) = \sum_{i=1}^{3} N_i(x,y) v_i = 0.0$$

$$\varepsilon_{xx}^{(2)} = \frac{\partial u^{(2)}}{\partial x} = -0.2$$

$$\varepsilon_{yy}^{(2)} = \frac{\partial v^{(2)}}{\partial y} = 0.0$$

$$\gamma_{xy}^{(2)} = \frac{\partial u^{(2)}}{\partial y} + \frac{\partial v^{(2)}}{\partial x} = -0.2$$

Strains are discontinuous along the element boundary

PRINCIPLE OF MINIMUM POTENTIAL ENERGY

- Strain Energy
  - energy that is stored in the structure due to the elastic deformation

$$U = \frac{1}{2}\iiint_{\text{volume}} \{\varepsilon\}^{\mathrm{T}}\{\sigma\}\,dV = \frac{h}{2}\iint_{\text{area}} \{\varepsilon\}^{\mathrm{T}}\{\sigma\}\,dA$$
$$= \frac{h}{2}\iint_{\text{area}} \{\varepsilon\}^{\mathrm{T}}[\mathbf{C}]\{\varepsilon\}\,dA$$

  - h: thickness, $[\mathbf{C}] = [\mathbf{C}_\sigma]$ for plane stress, and $[\mathbf{C}] = [\mathbf{C}_\varepsilon]$ for plane strain.
  - stress and strain are constant throughout the thickness.
  - The linear elastic relation $\{\sigma\} = [\mathbf{C}]\{\varepsilon\}$ has been used in the last relation.

PRINCIPLE OF MINIMUM POTENTIAL ENERGY *cont.*

- Potential Energy of Applied Loads
  - Force acting on a body reduces potential to do additional work.
  - Negative of product of the force and corresponding displacement
  - **Concentrated forces**

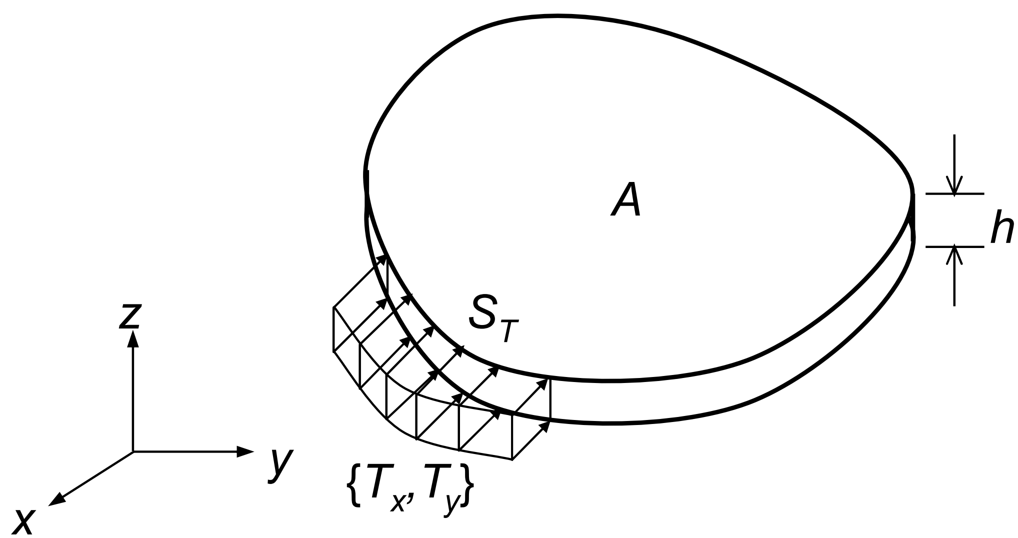
$$V = -\sum_{i=1}^{ND} F_i q_i$$

  - F_i and q_i are in the same direction
  - Reaction force does not have any potential (because $q_i = 0$)
  - **Distributed forces** (pressure load) acting on the edge

$$V = -h\int_{S_T} (T_x u + T_y v)\,dS$$

$$= -h\int_{S_T} [u \quad v]\begin{Bmatrix} T_x \\ T_y \end{Bmatrix} dS$$

$$\equiv -h\int_{S_T} \{\mathbf{u}\}^{\mathrm{T}}\{\mathbf{T}\}\,dS$$



PRINCIPLE OF MINIMUM POTENTIAL ENERGY *cont.*

- Total Potential Energy
  - Net energy contained in the structure
  - Sum of the strain energy and the potential energy of applied loads

$$\Pi = U + V$$

- Principle of Minimum Potential Energy
  - The structure is in equilibrium status when the potential energy has a minimum value.

$$\frac{\partial \Pi}{\partial \{\mathbf{u}\}} = 0 \Rightarrow \frac{\partial \Pi}{\partial u_1} = 0, \frac{\partial \Pi}{\partial u_2} = 0 \cdots \frac{\partial \Pi}{\partial u_N} = 0$$

$\Longrightarrow$ Finite Element Equation

CST ELEMENT *cont.*

- Strain Energy: $$U^{(e)} = \frac{h}{2}\iint_A \{\varepsilon\}^{\mathrm{T}}[\mathbf{C}]\{\varepsilon\}\,dA^{(e)}$$
$$= \frac{h}{2}\{\mathbf{q}^{(e)}\}^{\mathrm{T}}\iint_A [\mathbf{B}]^{\mathrm{T}}_{6\times3}[\mathbf{C}]_{3\times3}[\mathbf{B}]_{3\times6}\,dA\{\mathbf{q}^{(e)}\}$$
$$\equiv \frac{1}{2}\{\mathbf{q}^{(e)}\}^{\mathrm{T}}[\mathbf{k}^{(e)}]_{6\times6}\{\mathbf{q}^{(e)}\}$$
  - Element Stiffness Matrix: $\boxed{[\mathbf{k}^{(e)}] = hA[\mathbf{B}]^{\mathrm{T}}[\mathbf{C}][\mathbf{B}]}$
  - Different from the truss and beam elements, transformation matrix **[T]** is not required in the two-dimensional element because **[k]** is constructed in the global coordinates.
- The strain energy of the entire solid is simply the sum of the element strain energies

$$U = \sum_{e=1}^{NE} U^{(e)} = \frac{1}{2}\sum_{e=1}^{NE}\{\mathbf{q}^{(e)}\}^{\mathrm{T}}[\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\} \xrightarrow{\text{assembly}} U = \frac{1}{2}\{\mathbf{Q}_s\}^{\mathrm{T}}[\mathbf{K}_s]\{\mathbf{Q}_s\}$$

CST ELEMENT *cont.*

- Potential energy of concentrated forces at nodes

$$V = -\sum_{i=1}^{ND}(F_{ix}u_i + F_{iy}v_i) \equiv -\{\mathbf{Q}_s\}^T\{\mathbf{F}_N\}$$

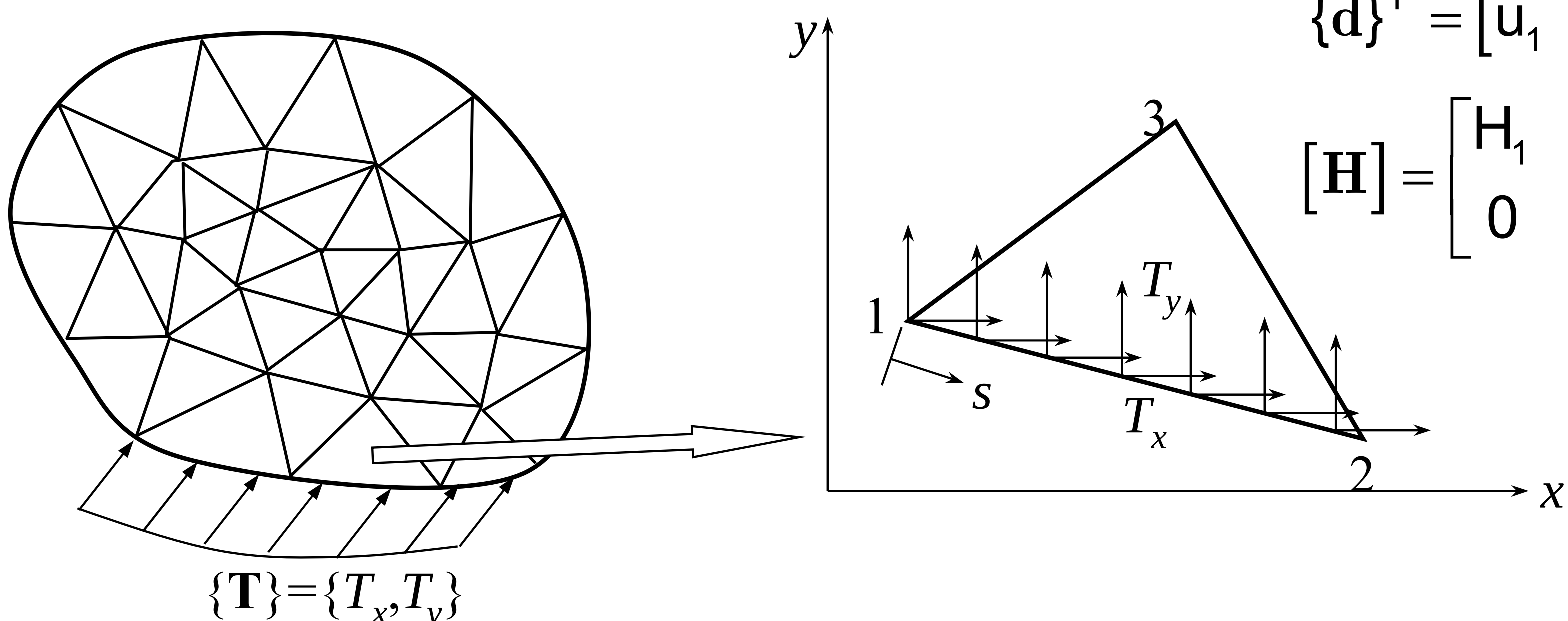
$$\{\mathbf{F}_N\} = [F_{1x} \quad F_{1y} \quad \cdots \quad F_{NDx} \quad F_{NDy}]^T$$

- Potential energy of distributed forces along element edges
  - Surface traction force {**T**} = [T_x, T_y]^T is applied on the element edge 1-2

$$V^{(e)} = -h\int_{S_T}\{\mathbf{u}(s)\}^T\{\mathbf{T}(s)\}\,ds = -\{\mathbf{d}\}^T h\int_{S_T}[\mathbf{H}(s)]^T\{\mathbf{T}(s)\}\,ds$$

$$\{\mathbf{d}\}^T = [u_1 \quad v_1 \quad u_2 \quad v_2]$$

$$[\mathbf{H}] = \begin{bmatrix} H_1 & 0 & H_2 & 0 \\ 0 & H_1 & 0 & H_2 \end{bmatrix}$$



CST ELEMENT *cont.*

- Rewrite with all 6 DOFs

$$V^{(e)} = -\{\mathbf{q}^{(e)}\}^T h \int_{S_T} [\mathbf{N}(s)]^T \{\mathbf{T}(s)\} ds = -\{\mathbf{q}^{(e)}\}^T \{\mathbf{f}_T^{(e)}\}$$

Work-equivalent nodal forces

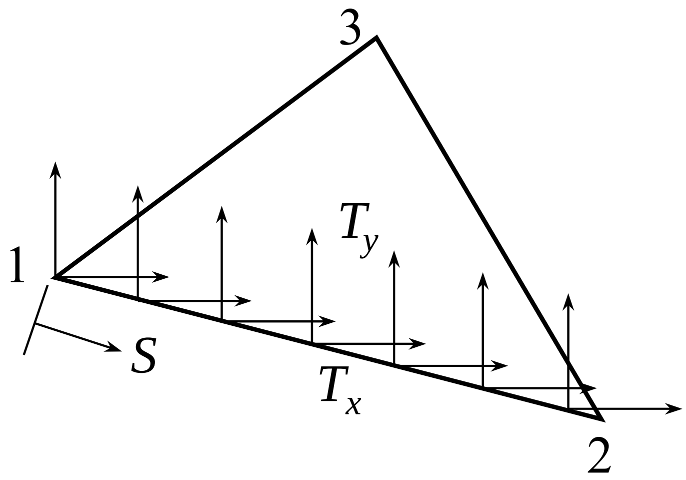
$$[\mathbf{N}] = \begin{bmatrix} \frac{l-s}{l} & 0 & \frac{s}{l} & 0 & 0 & 0 \\ 0 & \frac{l-s}{l} & 0 & \frac{s}{l} & 0 & 0 \end{bmatrix}$$

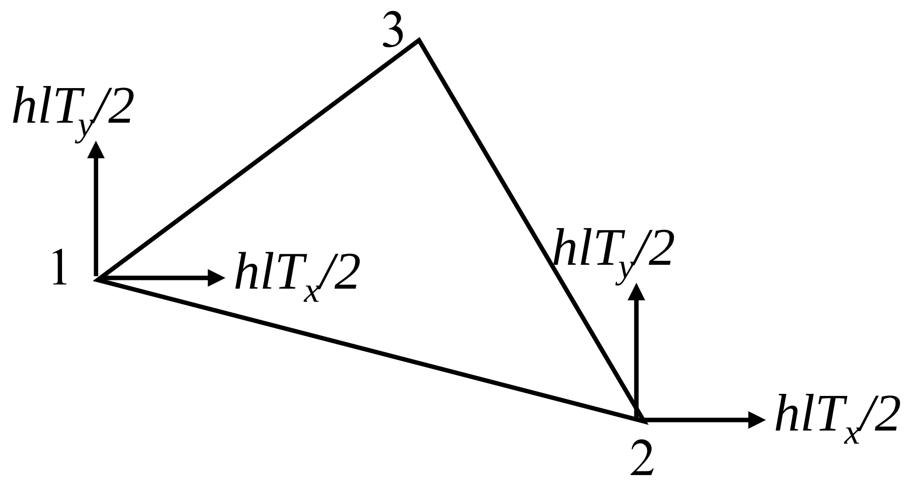
- Constant surface traction

$$\{\mathbf{f}_T^{(e)}\} = h\int_0^l [\mathbf{N}]^T \{\mathbf{T}\} ds = h\int_0^l \begin{bmatrix} (l-s)/l & 0 \\ 0 & (l-s)/l \\ s/l & 0 \\ 0 & s/l \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} ds = \frac{hl}{2} \begin{Bmatrix} T_x \\ T_y \\ T_x \\ T_y \\ 0 \\ 0 \end{Bmatrix}$$

Equally divided to two nodes

CST ELEMENT *cont.*





- Potential energy of distributed forces of all elements

$$V = -\sum_{e=1}^{NS} \{\mathbf{q}^{(e)}\}^T \{\mathbf{f}_T^{(e)}\} = -\{\mathbf{Q}_s\}^T \{\mathbf{F}_T\}$$

CST ELEMENT *cont.*

- Potential energy of body forces
  - distributed over the entire element (e.g. gravity or inertia forces).

$$V^{(e)} = -h\iint_A [u \quad v]\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dA = -\{\mathbf{q}^{(e)}\}^T h\iint_A [\mathbf{N}]^T dA \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}$$

$$\equiv -\{\mathbf{q}^{(e)}\}^T \{\mathbf{f}_b^{(e)}\}$$

$$\{\mathbf{f}_b^{(e)}\} = \frac{hA}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \frac{hA}{3}\begin{Bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{Bmatrix}$$

- Potential energy of body forces for all elements

$$V = -\sum_{e=1}^{NE} \{\mathbf{q}^{(e)}\}^T \{\mathbf{f}_b^{(e)}\} = -\{\mathbf{Q}_s\}^T \{\mathbf{F}_B\}$$

CST ELEMENT *cont.*

- Total Potential Energy

$$\Pi = U + V = \frac{1}{2}\{\mathbf{Q}_s\}^T[\mathbf{K}_s]\{\mathbf{Q}_s\} - \{\mathbf{Q}_s\}^T\{\mathbf{F}_N + \mathbf{F}_T + \mathbf{F}_B\}$$

- Principle of Minimum Potential Energy

$$\frac{\partial \Pi}{\partial \{\mathbf{Q}_s\}} = 0 \;\Rightarrow\; [\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_N + \mathbf{F}_T + \mathbf{F}_B\}$$

Finite Element Matrix Equation for CST Element

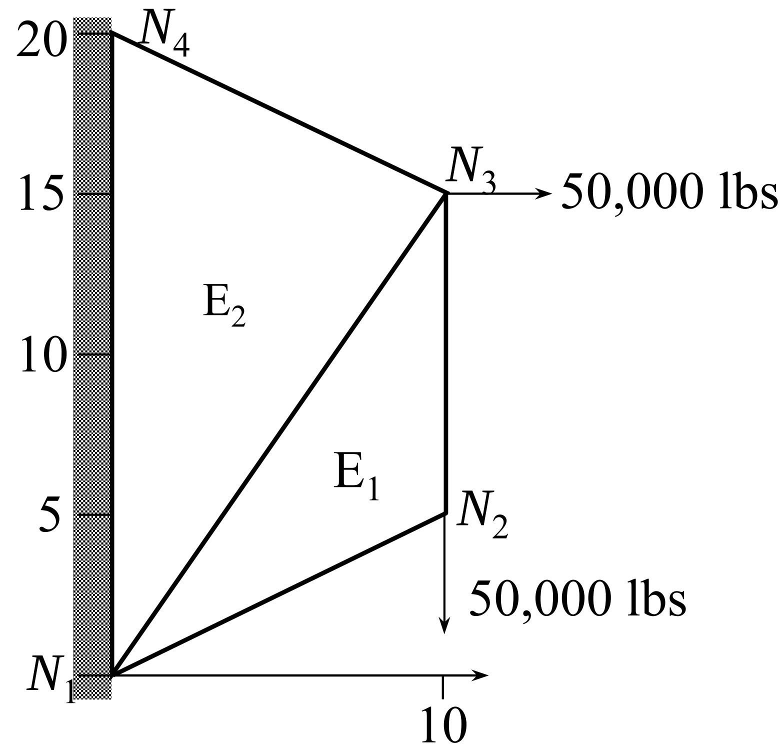
- Assembly and applying boundary conditions are identical to other elements (beam and truss).
- Stress and Strain Calculation
  - Nodal displacement $\{\mathbf{q}^{(e)}\}$ for the element of interest needs to be extracted

$$\{\varepsilon\} = [\mathbf{B}]\{\mathbf{q}^{(e)}\} \qquad \{\sigma\} = [\mathbf{C}]\{\varepsilon\} = [\mathbf{C}][\mathbf{B}]\{\mathbf{q}^{(e)}\}$$

Stress and strain are constant for CST element

EXAMPLE

- Cantilevered Plate
  - Thickness h = 0.1 in, E = 30×10^6 psi and v = 0.3.
- Element 1
  - Area = 0.5×10×10 = 50.



$$x_1 = 0,\ y_1 = 0 \qquad x_2 = 10,\ y_2 = 5 \qquad x_3 = 10,\ y_3 = 15$$

$$b_1 = y_2 - y_3 = -10 \qquad b_2 = y_3 - y_1 = 15 \qquad b_3 = y_1 - y_2 = -5$$

$$c_1 = x_3 - x_2 = 0 \qquad c_2 = x_1 - x_3 = -10 \qquad c_3 = x_2 - x_1 = 10$$

EXAMPLE *cont.*

- Matrix [B]

$$[\mathbf{B}] = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} -10 & 0 & 15 & 0 & -5 & 0 \\ 0 & 0 & 0 & -10 & 0 & 10 \\ 0 & -10 & -10 & 15 & 10 & -5 \end{bmatrix}$$

- Plane Stress Condition

$$[\mathbf{C}_\sigma] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} = 3.297\times 10^7\begin{bmatrix} 1 & .3 & 0 \\ .3 & 1 & 0 \\ 0 & 0 & .35 \end{bmatrix}$$

EXAMPLE *cont.*

- Stiffness Matrix for Element 1

$$[\mathbf{k}^{(1)}] = hA[\mathbf{B}]^T[\mathbf{C}_\sigma][\mathbf{B}] = 3.297\times10^6\begin{bmatrix} .5 & 0. & -.75 & .15 & .25 & -.15 \\ & .175 & .175 & -.263 & -.175 & .088 \\ & & 1.3 & -.488 & -.55 & .313 \\ & & & .894 & .338 & -.631 \\ & & & & .3 & -.163 \\ & & & & & .544 \end{bmatrix}$$

- Element 2: Nodes 1-3-4

$$x_1 = 0,\ y_1 = 0 \qquad x_2 = 10,\ y_2 = 15 \qquad x_3 = 0,\ y_3 = 20$$

$$b_1 = y_2 - y_3 = -5 \qquad b_2 = y_3 - y_1 = 20 \qquad b_3 = y_1 - y_2 = -15$$

$$c_1 = x_3 - x_2 = -10 \qquad c_2 = x_1 - x_3 = 0 \qquad c_3 = x_2 - x_1 = 10$$

EXAMPLE *cont.*

- Matrix [B]

$$[\mathbf{B}] = \frac{1}{200}\begin{bmatrix} -5 & 0 & 20 & 0 & -15 & 0 \\ 0 & -10 & 0 & 0 & 0 & 10 \\ -10 & -5 & 0 & 20 & 10 & -15 \end{bmatrix}$$

- Stiffness Matrix

$$[\mathbf{k}^{(2)}] = 3.297\times 10^{6}\begin{bmatrix} .15 & .081 & -.25 & -.175 & .1 & .094 \\ & .272 & -.15 & -.088 & .069 & -.184 \\ & & 1. & 0. & -.75 & .15 \\ & & & .35 & .175 & -.263 \\ & & & & .65 & -.244 \\ & & & & & .447 \end{bmatrix}$$

EXAMPLE *cont.*

- Assembly

$$3.297\times10^{6}\begin{bmatrix} .65 & .081 & -.75 & .15 & .0 & -.325 & .1 & .094 \\ & .447 & .175 & -.263 & -.325 & .0 & .069 & -.184 \\ & & 1.3 & -.488 & -.55 & .313 & .0 & .0 \\ & & & .894 & .338 & -.631 & .0 & .0 \\ & & & & 1.3 & -.163 & -.75 & .15 \\ & & & & & .894 & .175 & -.263 \\ & \text{Symmetric} & & & & & .65 & -.244 \\ & & & & & & & .447 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}=\begin{Bmatrix} R_{x1} \\ R_{y1} \\ 0 \\ -50{,}000 \\ 50{,}000 \\ 0 \\ R_{x4} \\ R_{y4} \end{Bmatrix}$$

- R_{x1}, R_{y1}, R_{x4}, and R_{y4} are unknown reactions at nodes 1 and 4
- displacement boundary condition $u_1 = v_1 = u_4 = v_4 = 0$

EXAMPLE *cont.*

- Reduced Matrix Equation and Solution

$$3.297\times 10^{6}\begin{bmatrix} 1.3 & -.488 & -.55 & .313 \\ & .894 & .338 & -.631 \\ & & 1.3 & -.163 \\ & & & .894 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ -50,000 \\ 50,000 \\ 0 \end{Bmatrix}$$

$$\begin{aligned} u_2 &= -2.147\times 10^{-3} \\ v_2 &= -4.455\times 10^{-2} \\ u_3 &= \ \ 1.891\times 10^{-2} \\ v_3 &= -2.727\times 10^{-2} \end{aligned}$$

EXAMPLE *cont.*

- Element Results
  - Element 1

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{100} \begin{bmatrix} -10 & 0 & 15 & 0 & -5 & 0 \\ 0 & 0 & 0 & -10 & 0 & 10 \\ 0 & -10 & -10 & 15 & 10 & -5 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -2.147\times10^{-3} \\ -4.455\times10^{-2} \\ 1.891\times10^{-2} \\ -2.727\times10^{-2} \end{Bmatrix} = \begin{Bmatrix} -1.268\times10^{-3} \\ 1.727\times10^{-3} \\ -3.212\times10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = 3.297\times10^{7} \begin{bmatrix} 1 & .3 & 0 \\ .3 & 1 & 0 \\ 0 & 0 & .35 \end{bmatrix} \begin{Bmatrix} -1.268\times10^{-3} \\ 1.727\times10^{-3} \\ -3.212\times10^{-3} \end{Bmatrix} = \begin{Bmatrix} -24,709 \\ 44,406 \\ -37,063 \end{Bmatrix} \text{psi}$$

EXAMPLE *cont.*

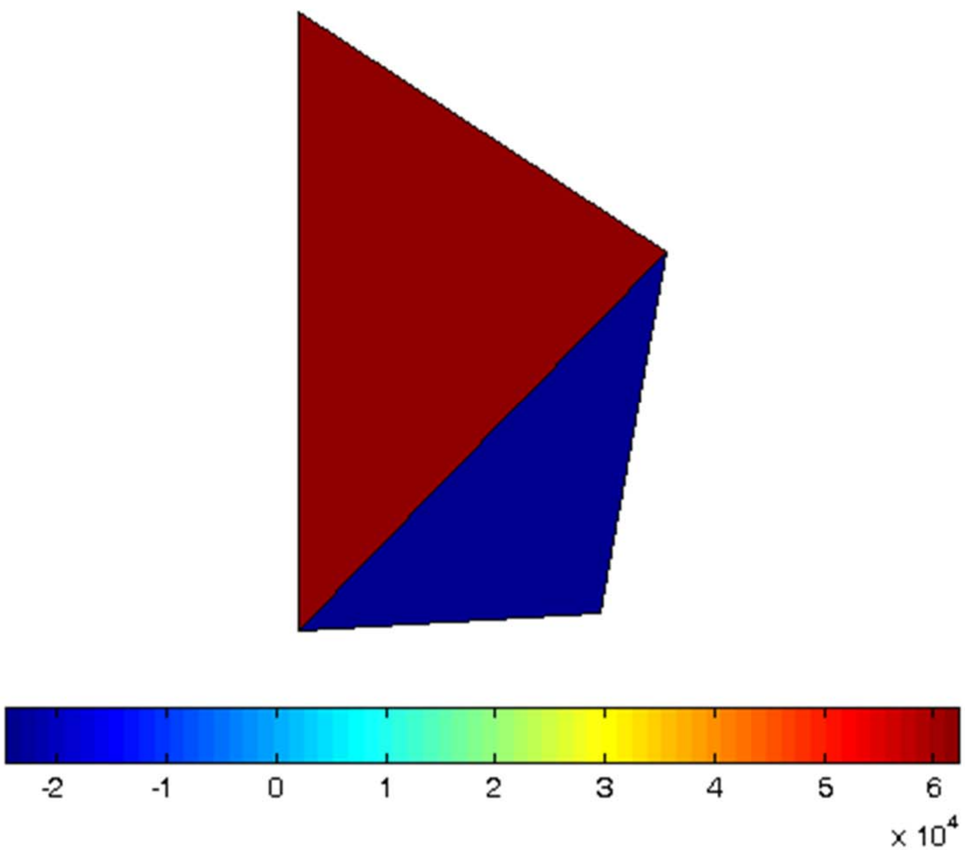
- Element Results
  - Element 2

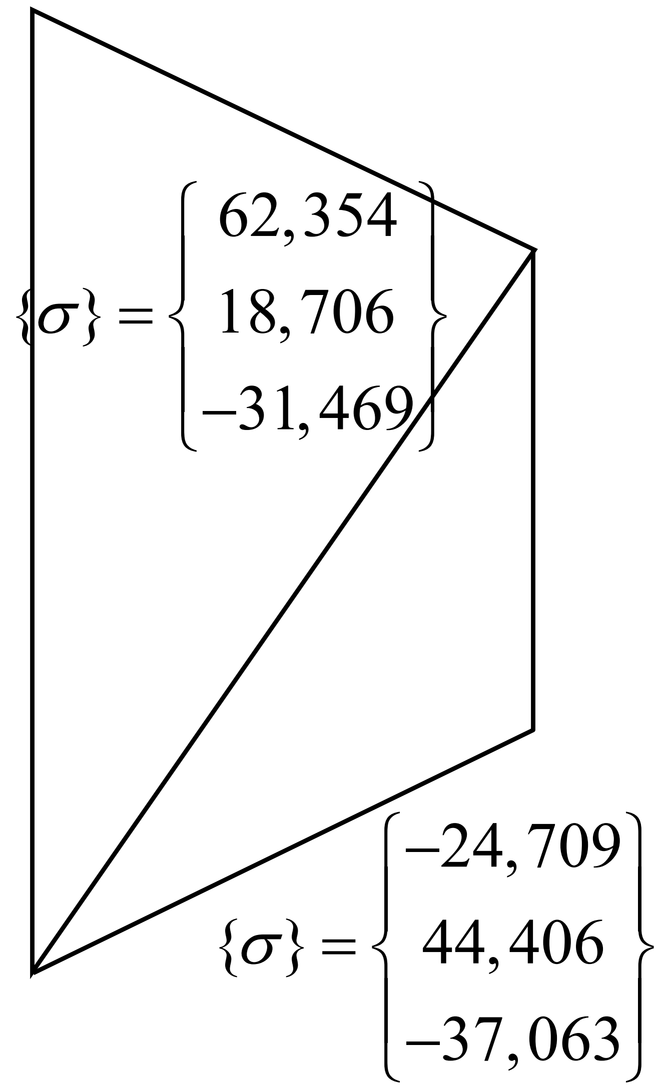
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{200} \begin{bmatrix} -5 & 0 & 20 & 0 & -15 & 0 \\ 0 & -10 & 0 & 0 & 0 & 10 \\ -10 & -5 & 0 & 20 & 10 & -15 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1.891 \times 10^{-2} \\ -2.727 \times 10^{-2} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.891 \times 10^{-3} \\ 0 \\ -2.727 \times 10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = 3.297 \times 10^{7} \begin{bmatrix} 1 & .3 & 0 \\ .3 & 1 & 0 \\ 0 & 0 & .35 \end{bmatrix} \begin{Bmatrix} 1.891 \times 10^{-3} \\ 0 \\ -2.727 \times 10^{-3} \end{Bmatrix} = \begin{Bmatrix} 62{,}354 \\ 18{,}706 \\ -31{,}469 \end{Bmatrix} \text{psi}$$

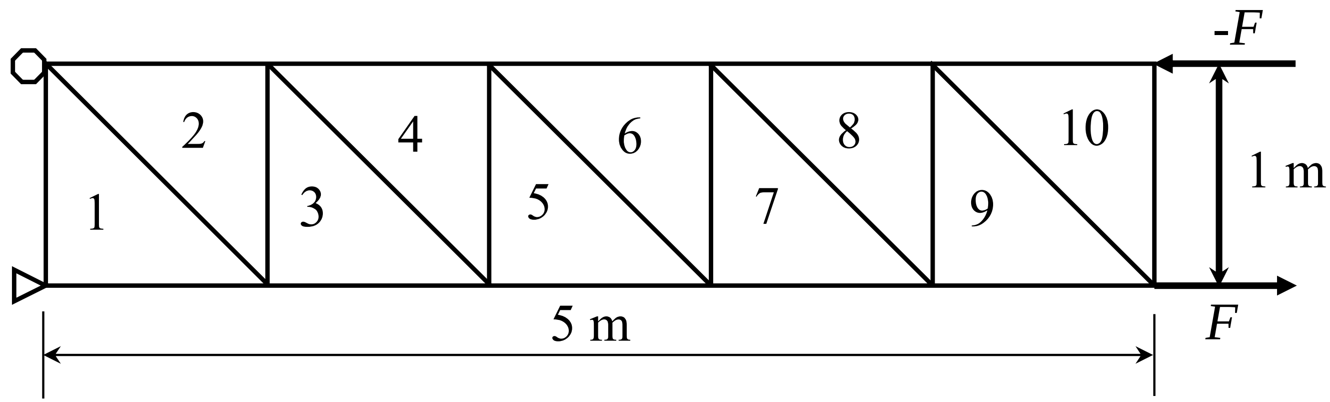
DISCUSSION

- These stresses are constant over respective elements.
- large discontinuity in stresses across element boundaries



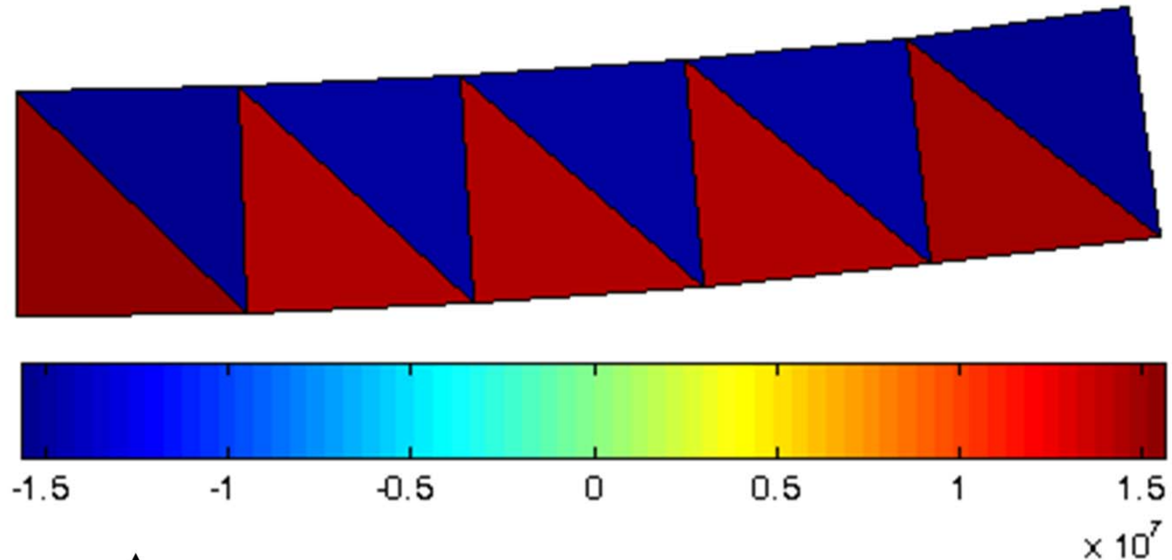


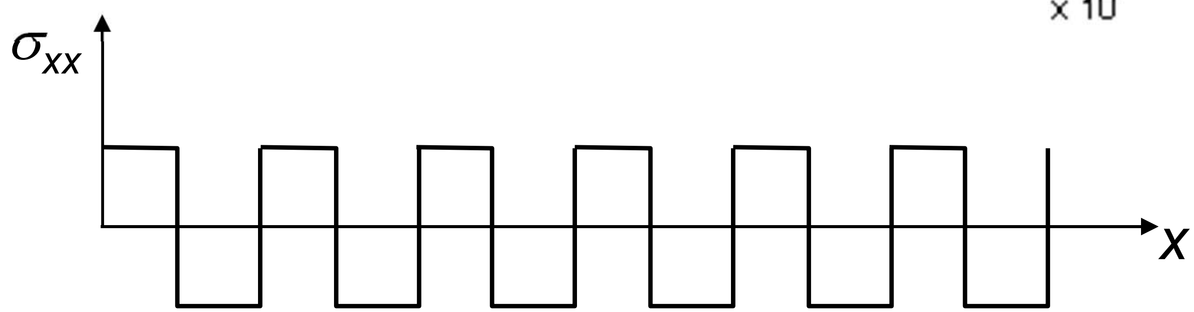
BEAM BENDING EXAMPLE



- σ_{xx} is constant along the x-axis and linear along y-axis
- Exact Solution: σ_{xx} = 60 MPa
- Max deflection v_{max} = 0.0075 m

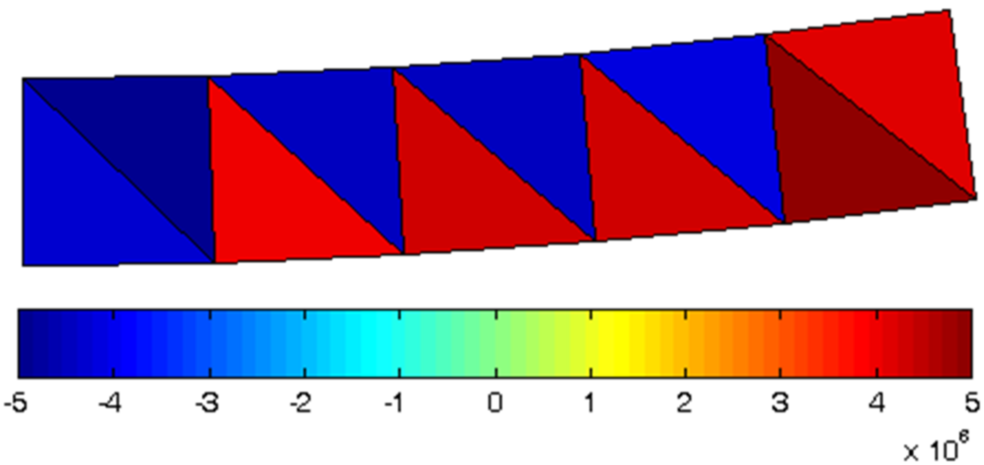
Max v = 0.0018



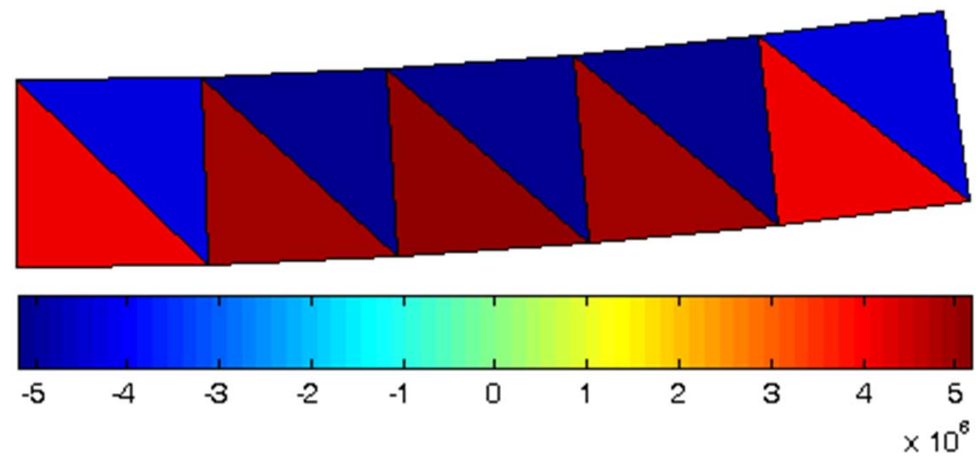


BEAM BENDING EXAMPLE *cont*.

- y-normal stress and shear stress are supposed to be zero.



σ_{yy} Plot



τ_{xy} Plot

CST Element in Bending

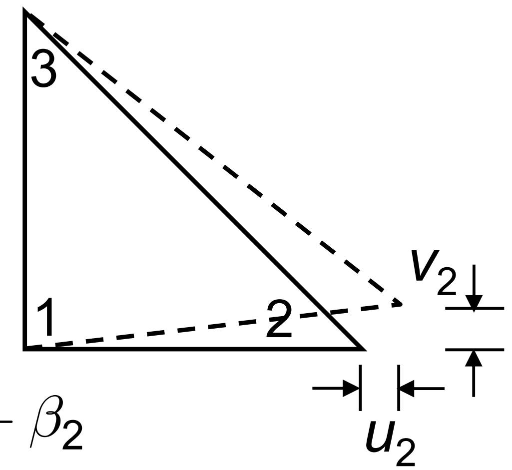
- Discussions
  - CST element performs well when strain gradient is small.
  - In pure bending problem, σ_{xx} in the neutral axis should be zero. Instead, CST elements show oscillating pattern of stress.
  - CST elements predict stress and deflection about ¼ of the exact values.
    - Strain along y-axis is supposed to be linear. But, CST elements can only have constant strain in y-direction.
    - CST elements also have spurious shear strain.

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v(x,y) = \beta_1 + \beta_2 x + \beta_3 y$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \alpha_2$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \beta_3$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \alpha_3 + \beta_2$$



How can we improve accuracy?
What direction?

CST Element in Bending

- When u_2 & v_2 are not zero

$$\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} f_1 & f_2 & f_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ 0\end{Bmatrix} = \frac{u_2}{2A}\begin{Bmatrix}f_2\\ b_2\\ c_2\end{Bmatrix}$$

$$\begin{Bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} f_1 & f_2 & f_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\begin{Bmatrix}0\\ v_2\\ 0\end{Bmatrix} = \frac{v_2}{2A}\begin{Bmatrix}f_2\\ b_2\\ c_2\end{Bmatrix}$$

$$u(x,y) = \frac{u_2}{2A}(f_2 + b_2 x + c_2 y)$$

$$v(x,y) = \frac{v_2}{2A}(f_2 + b_2 x + c_2 y)$$

$$b_2 = y_3 - y_1 = h$$

$$c_2 = x_1 - x_3 = 0$$

- Strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \alpha_2 = \frac{b_2}{2A}u_2 = \frac{hu_2}{2A}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \beta_3 = \frac{c_2}{2A}v_2 = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{c_2}{2A}u_2 + \frac{b_2}{2A}v_2 = \frac{hv_2}{2A}$$

$\Rightarrow$

Stress

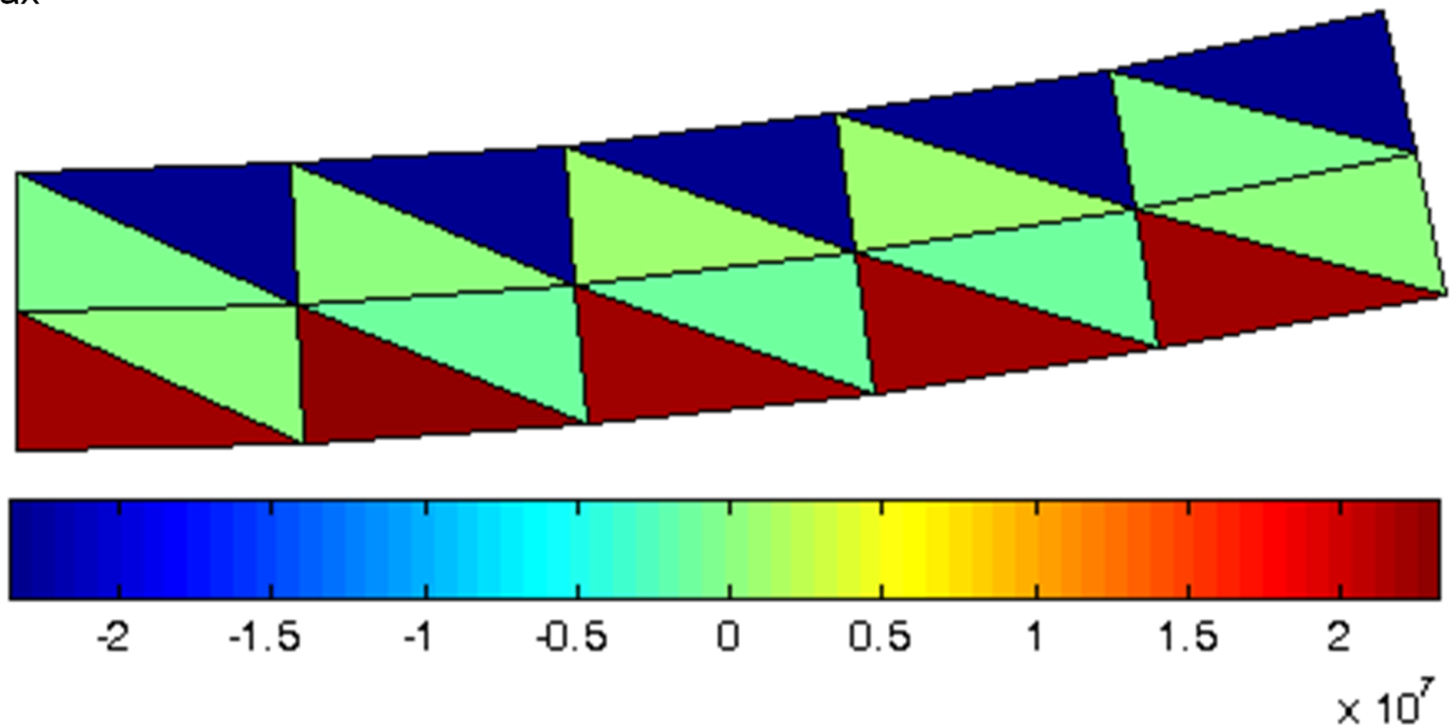
$$\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx} = \frac{Ehu_2}{2A(1-\nu^2)}$$

$$\sigma_{yy} = \frac{\nu E}{1-\nu^2}\varepsilon_{xx} = \frac{\nu Ehu_2}{2A(1-\nu^2)}$$

$$\tau_{xy} = G\gamma_{xy} = \frac{Ehv_2}{4A(1+\nu)}$$

CST ELEMENT *cont.*

- Two-Layer Model
  - $\sigma_{xx} = 2.32 \times 10^7$
  - $v_{max} = 0.0028$



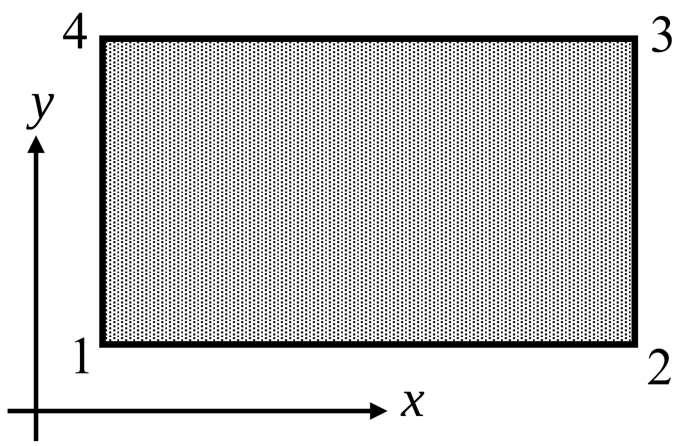
RECTANGULAR ELEMENT

RECTANGULAR ELEMENT

- Each edge is parallel to the coordinate direction (not practical)
- Lagrange interpolation for shape function calculation
- Interpolation:

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$
$$v = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$

$$\begin{cases} u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 + \alpha_4 x_1 y_1 \\ u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 + \alpha_4 x_2 y_2 \\ u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 + \alpha_4 x_3 y_3 \\ u_4 = \alpha_1 + \alpha_2 x_4 + \alpha_3 y_4 + \alpha_4 x_4 y_4 \end{cases}$$

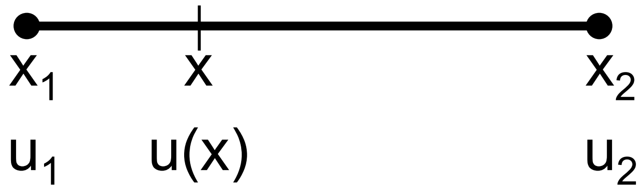


?

$$u(x,y) = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

LAGRANGE INTERPOLATION

- Interpolation



$$u(x) = N_1(x)u_1 + N_2(x)u_2$$

$$N_1(x) = \frac{x - x_2}{x_1 - x_2}, \; N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

$$u(x) = \frac{l - x}{l}u_1 + \frac{x}{l}u_2$$

RECTANGULAR ELEMENT *cont.*

- Lagrange Interpolation
  - Along edge 1-2, $y = y_1$ (constant)

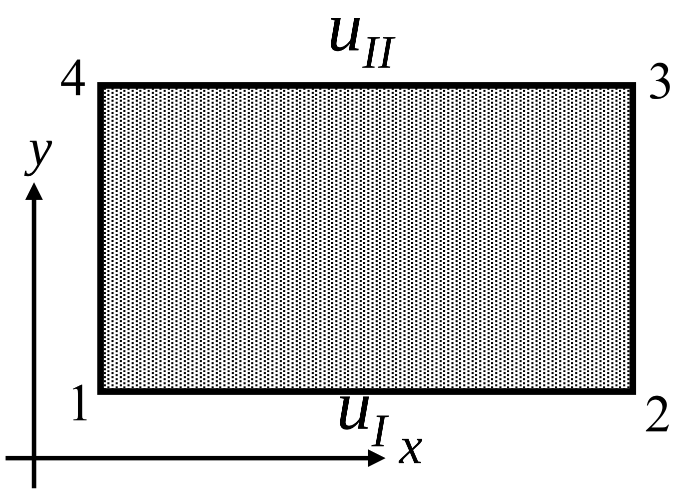
$$u_I(x,y_1) = [n_1(x) \quad n_2(x)]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$n_1(x) = \frac{x - x_2}{x_1 - x_2}, \quad n_2(x) = \frac{x - x_1}{x_2 - x_1}$$

  - Along edge 4-3, $y = y_3$ (constant)

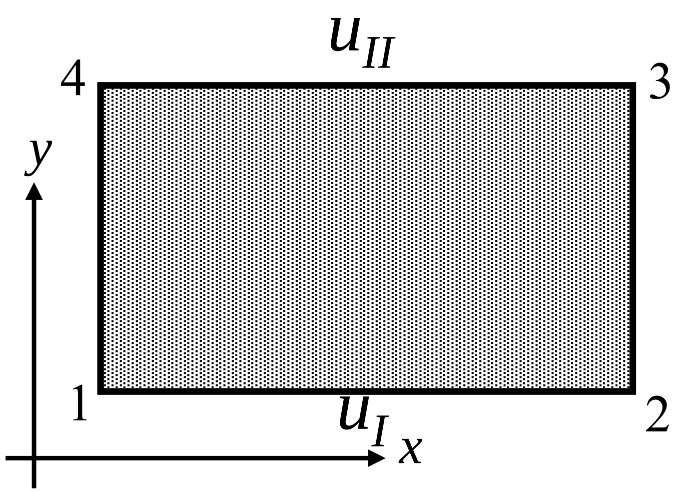
$$u_{II}(x,y_3) = [n_4(x) \quad n_3(x)]\begin{Bmatrix} u_4 \\ u_3 \end{Bmatrix}$$

$$n_4(x) = \frac{x - x_3}{x_4 - x_3}, \quad n_3(x) = \frac{x - x_4}{x_3 - x_4}$$



RECTANGULAR ELEMENT *cont.*

- Lagrange Interpolation
  - Y-direction



$$u(x,y) = [n_1(y) \quad n_4(y)] \begin{Bmatrix} u_I(x, y_1) \\ u_{II}(x, y_3) \end{Bmatrix}$$

$$n_1(y) = \frac{y - y_4}{y_1 - y_4}, \quad n_4(y) = \frac{y - y_1}{y_4 - y_1}$$

  - Combine together

$$u(x,y) = [n_1(y) \quad n_4(y)] \begin{Bmatrix} [n_1(x) \quad n_2(x)] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ [n_4(x) \quad n_3(x)] \begin{Bmatrix} u_4 \\ u_3 \end{Bmatrix} \end{Bmatrix}$$

RECTANGULAR ELEMENT *cont.*

- Lagrange Interpolation

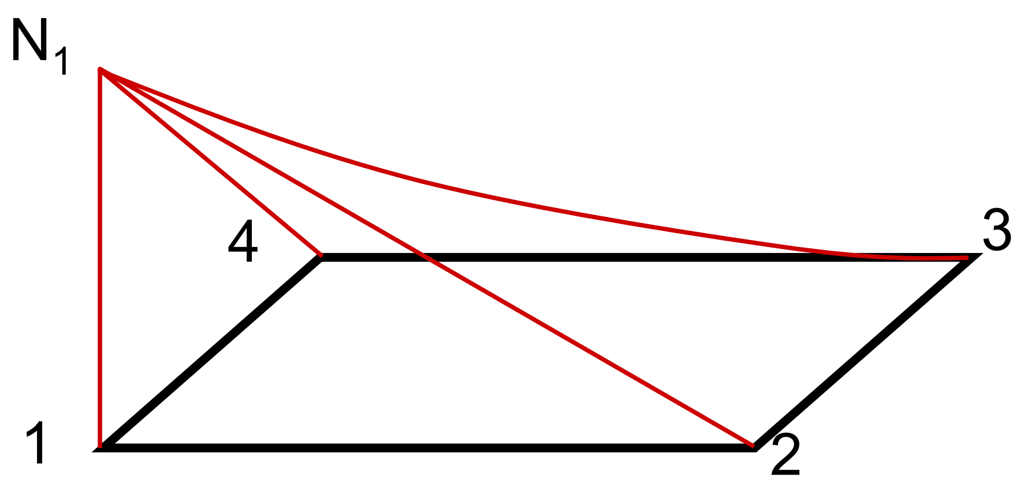
$$u(x,y) = [n_1(x)n_1(y) \quad n_2(x)n_1(y) \quad n_3(x)n_4(y) \quad n_4(x)n_4(y)]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\begin{cases} N_1 \equiv n_1(x)n_1(y) = \dfrac{1}{A}(x_3 - x)(y_3 - y) \\ N_2 \equiv n_2(x)n_1(y) = -\dfrac{1}{A}(x_1 - x)(y_3 - y) \\ N_3 \equiv n_3(x)n_4(y) = \dfrac{1}{A}(x_1 - x)(y_1 - y) \\ N_4 \equiv n_4(x)n_4(y) = -\dfrac{1}{A}(x_3 - x)(y_1 - y) \end{cases}$$

Bi-linear interpolation

RECTANGULAR ELEMENT *cont.*

- Shape functions for rectangular elements are product of Lagrange interpolations in the two coordinate directions.
- Note that $N_1(x, y)$ is:
  - 1 at node 1 and 0 at other nodes.
  - Linear function of x along edge 1-2 and linear function of y along edge 1-4.
  - Zero along edge 2-3 and 3-4.
- Other shape functions have similar behavior.



RECTANGULAR ELEMENT *cont.*

- Displacement interpolation
  - Same interpolation for both u and v.

$$\{\mathbf{u}\} \equiv \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

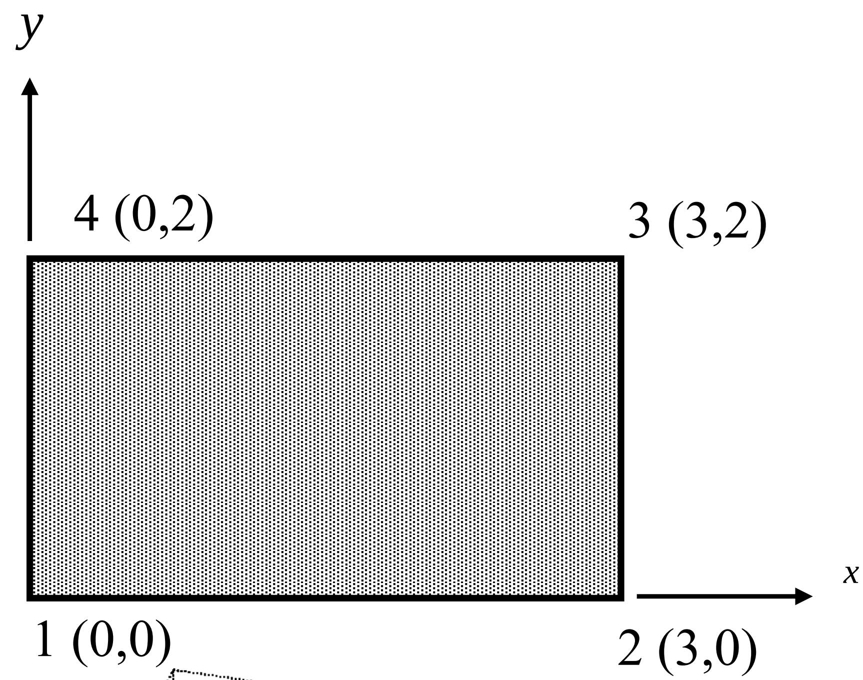
$$\boxed{\{\mathbf{u}\} = [\mathbf{N}]_{2\times 8}\{\mathbf{q}\}_{8\times 1}}$$

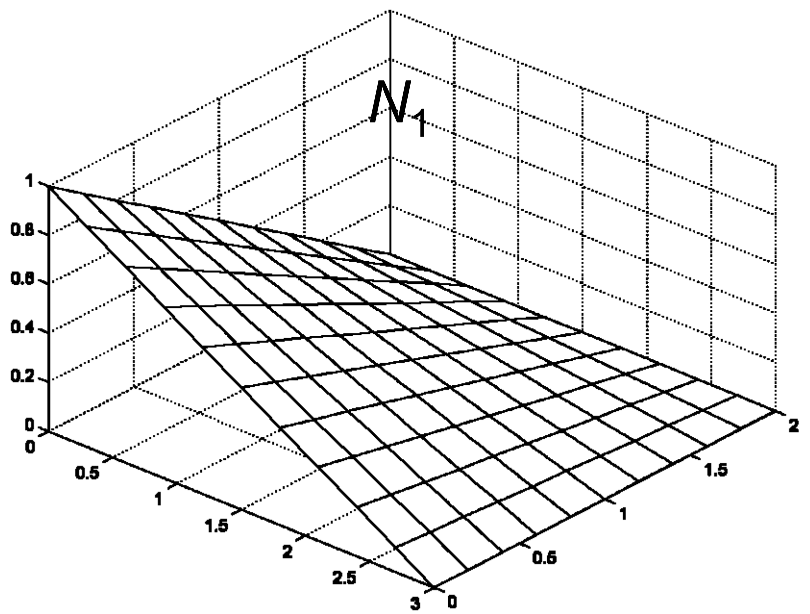
EXAMPLE 8.2

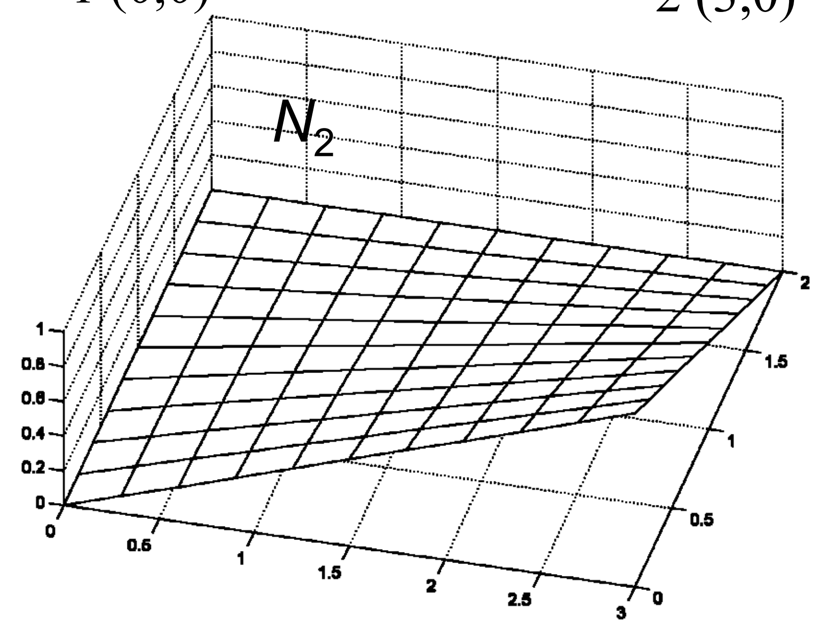
- Shape Functions

$$N_1 = \frac{(x-3)(y-2)}{6} \qquad N_2 = \frac{-x(y-2)}{6}$$

$$N_3 = \frac{xy}{6} \qquad N_4 = \frac{-y(x-3)}{6}$$







RECTANGULAR ELEMENT *cont.*

- Strain-displacement relation
  - Similar to CST element

$$\varepsilon_{xx} \equiv \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\sum_{i=1}^{4} N_i(x,y)u_i\right) = \sum_{i=1}^{4} \frac{\partial N_i}{\partial x} u_i$$

$$\{\varepsilon\} = \frac{1}{A}\begin{bmatrix} y-y_3 & 0 & y_3-y & 0 & y-y_1 & 0 & y_1-y & 0 \\ 0 & x-x_3 & 0 & x_1-x & 0 & x-x_1 & 0 & x_3-x \\ x-x_3 & y-y_3 & x_1-x & y_3-y & x-x_1 & y-y_1 & x_3-x & y_1-y \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\equiv [\mathbf{B}]\{\mathbf{q}\}$$

  - [**B**] is a linear function of *x* and *y*.
  - Strain will change linearly within the element (not completely linear in both directions)

RECTANGULAR ELEMENT *cont.*

- Element stiffness matrix (from strain energy)

$$U^{(e)} = \frac{h}{2}\iint_A \{\varepsilon\}^T[\mathbf{C}]\{\varepsilon\}\,dA^{(e)} = \frac{h}{2}\{\mathbf{q}^{(e)}\}^T \iint_A [\mathbf{B}]^T_{8\times3}[\mathbf{C}]_{3\times3}[\mathbf{B}]_{3\times8}\,dA\{\mathbf{q}^{(e)}\}$$

$$\equiv \frac{1}{2}\{\mathbf{q}^{(e)}\}^T[\mathbf{k}^{(e)}]_{8\times8}\{\mathbf{q}^{(e)}\}$$

  - Generally needs numerical integration
  - A square, plane-stress, rectangular element:

$$[\mathbf{k}^{(e)}] = \frac{Eh}{1-\nu^2}\begin{bmatrix}
\frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} \\
\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} \\
-\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} \\
\frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} \\
\frac{-3+\nu}{12} & -\frac{1+\nu}{8} & \frac{\nu}{6} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{-1+3\nu}{8} \\
-\frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{3-\nu}{6} & \frac{1-3\nu}{8} & \frac{\nu}{6} \\
\frac{\nu}{6} & \frac{-1+3\nu}{8} & \frac{-3+\nu}{12} & \frac{1+\nu}{8} & -\frac{3+\nu}{12} & \frac{1-3\nu}{8} & \frac{3-\nu}{6} & -\frac{1+\nu}{8} \\
\frac{1-3\nu}{8} & -\frac{3+\nu}{12} & \frac{1+\nu}{8} & \frac{-3+\nu}{12} & \frac{-1+3\nu}{8} & \frac{\nu}{6} & -\frac{1+\nu}{8} & \frac{3-\nu}{6}
\end{bmatrix}$$

RECTANGULAR ELEMENT *cont.*

- Nodal and distributed forces are the same with CST element
- Body force (constant body force **b** = $\{b_x, b_y\}^T$)

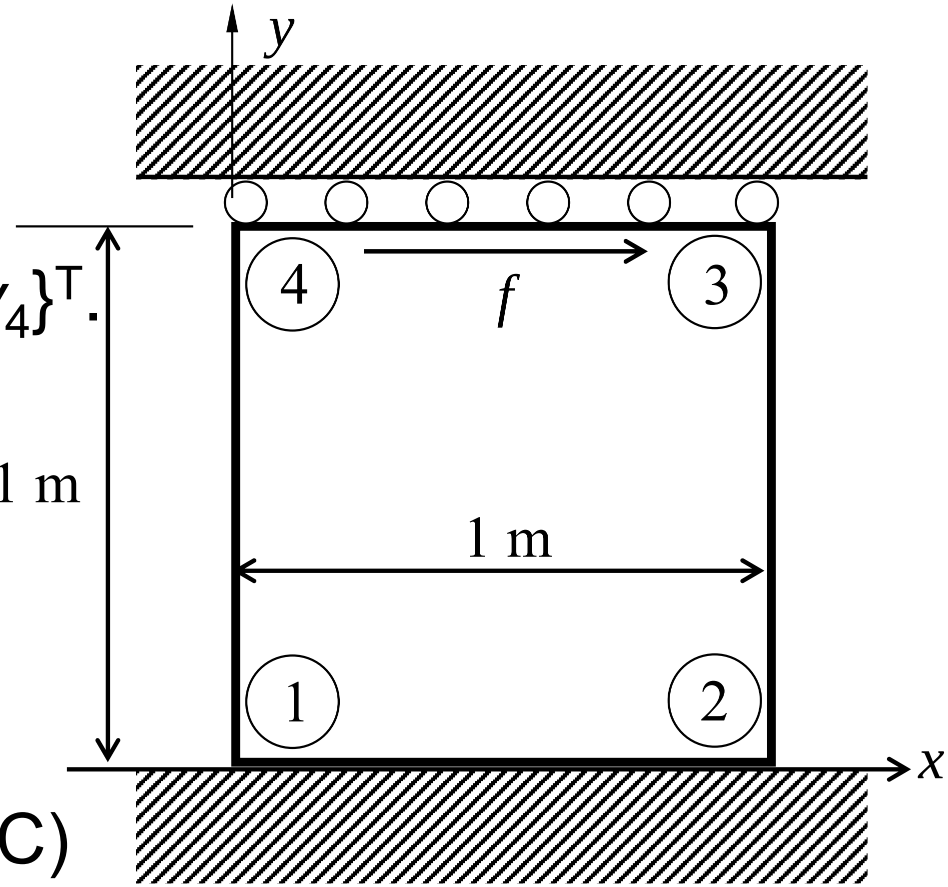
$$V^{(e)} = -h\iint_A [u \quad v]\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dA = -\{\mathbf{q}^{(e)}\}^T h \iint_A [\mathbf{N}]^T dA \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}$$

$$\equiv \{\mathbf{q}^{(e)}\}^T \{\mathbf{f}_b^{(e)}\}$$

$$\{\mathbf{f}_b^{(e)}\} = \frac{hA}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \frac{hA}{4}\begin{Bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{Bmatrix}$$

Equally divide the total magnitude of the body force to the four nodes

EXAMPLE – SIMPLE SHEAR



- E = 10 GPa, v = 0.25, h=0.1m
- F = 100 kN/m^2
- $\{\mathbf{Q}_s\} = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}^T$.
- Non-zero DOFs: u_3 and u_4.
- Stiffness matrix

$$[\mathbf{K}] = \frac{Eh}{1-\nu^2}\begin{bmatrix} \frac{3-\nu}{6} & -\frac{3+\nu}{12} \\ -\frac{3+\nu}{12} & \frac{3-\nu}{6} \end{bmatrix} \begin{matrix} u_3 \\ u_4 \end{matrix}$$

- FEM equation (after applying BC)

$$10^8 \begin{bmatrix} 4.88 & -2.88 \\ -2.88 & 4.88 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 5{,}000 \\ 5{,}000 \end{Bmatrix}$$

- Nodal displacements

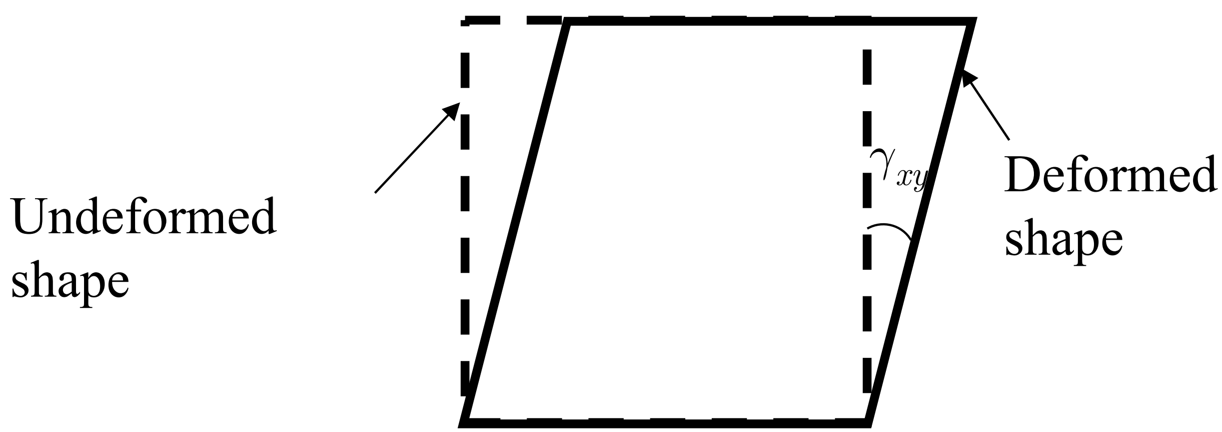
$$u_3 = u_4 = 0.025\text{mm}$$

EXAMPLE – SIMPLE SHEAR *cont.*

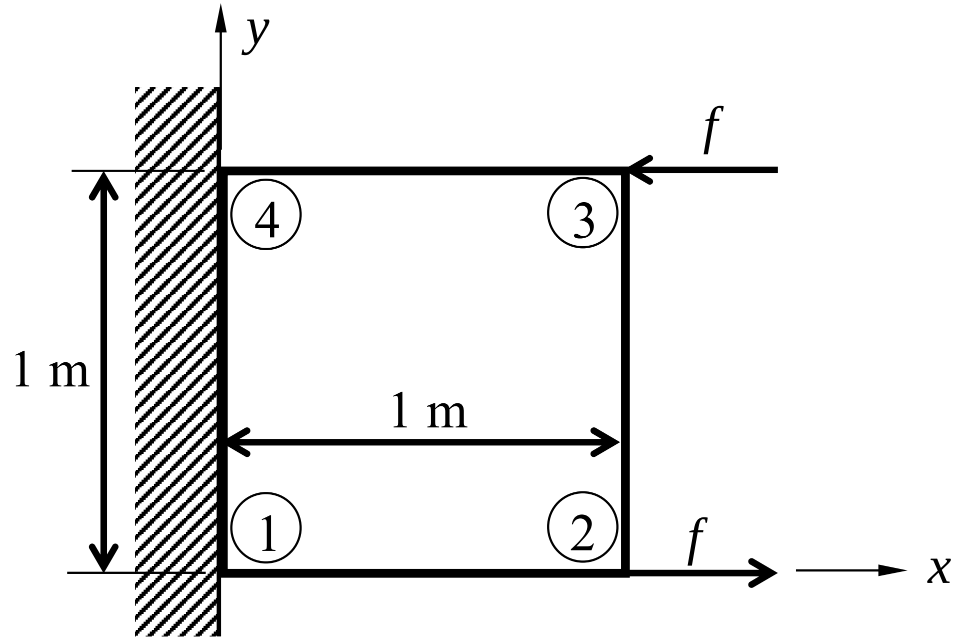
- Strain & Stress

$$\{\varepsilon\} = \begin{bmatrix} y-1 & 0 & 1-y & 0 & y & 0 & -y & 0 \\ 0 & x-1 & 0 & -x & 0 & x & 0 & 1-x \\ x-1 & y-1 & -x & 1-y & x & y & 1-x & -y \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2.5\times10^{-5} \\ 0 \\ 2.5\times10^{-5} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 2.5\times10^{-5} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{10^{10}}{1-0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 2.5\times10^{-5} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10^5 \end{Bmatrix} \text{Pa}$$



EXAMPLE – PURE BENDING



- Couple M = 100 kN.m
- Analytical solution

$$(\sigma_{xx})_{max} = -\frac{M(-\frac{h}{2})}{I} = 6.0\,\text{MPa}$$

$$\sigma_{xx} = 6.0(1-2y)\,\text{MPa}$$

- FEM solution
  - Non-zero DOFs: u_2, v_2, u_3, and v_3.

$$10^8\begin{bmatrix} 4.89 & -1.67 & 0.44 & -0.33 \\ -1.67 & 4.89 & 0.33 & -2.89 \\ 0.44 & 0.33 & 4.89 & 1.67 \\ -0.33 & -2.89 & 1.67 & 4.89 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 100{,}000 \\ 0 \\ -100{,}000 \\ 0 \end{Bmatrix}$$

$$u_2 = 0.4091\,\text{mm}, \qquad v_2 = 0.4091\,\text{mm}$$

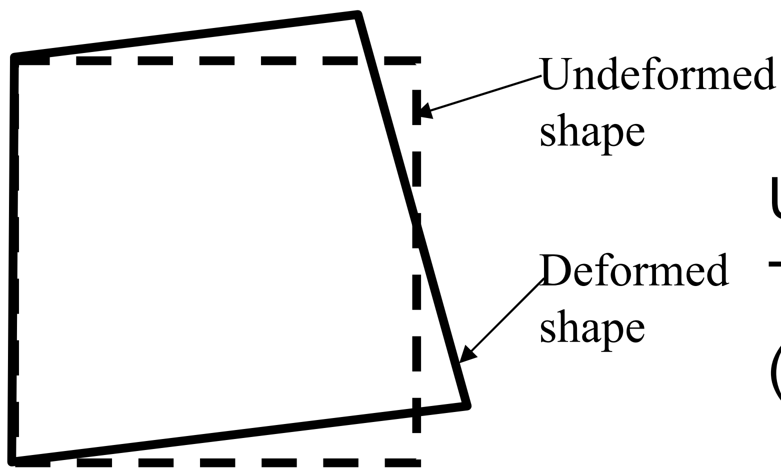
$$u_3 = -0.4091\,\text{mm}, \qquad v_3 = 0.4091\,\text{mm}$$

EXAMPLE – PURE BENDING *cont.*

- Strain & Stress

$$\{\varepsilon\} = \begin{bmatrix} y-1 & 0 & 1-y & 0 & y & 0 & -y & 0 \\ 0 & x-1 & 0 & -x & 0 & x & 0 & 1-x \\ x-1 & y-1 & -x & 1-y & x & y & 1-x & -y \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.41 \\ 0.41 \\ -0.41 \\ 0.41 \\ 0 \\ 0 \end{Bmatrix} \times 10^{-3} = \begin{Bmatrix} 0.41\times10^{-3}(1-2y) \\ 0 \\ 0.41\times10^{-3}(1-2x) \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{10^{10}}{1-0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \begin{Bmatrix} 0.41\times10^{-3}(1-2y) \\ 0 \\ 0.41\times10^{-3}(1-2x) \end{Bmatrix} = \begin{Bmatrix} 4.4(1-2y) \\ 1.1(1-2y) \\ 1.6(1-2x) \end{Bmatrix} \text{MPa}$$



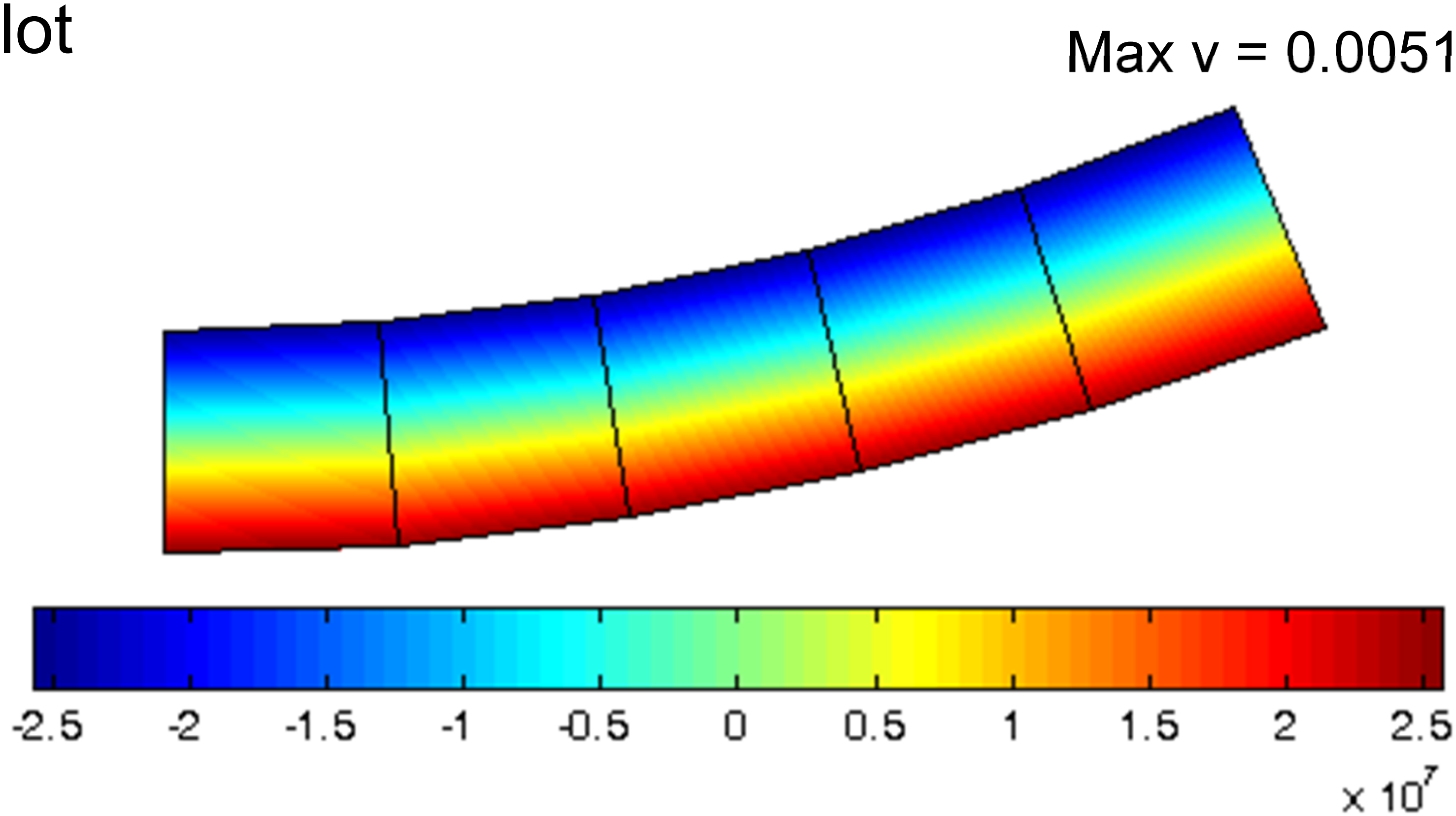
Unable to make curvature

Trapezoidal shape -> non-zero shear stress

$(\sigma_{xx})_{max} / (\sigma_{xx})_{exact}$ = 4.364/6.0 (73%)

BEAM BENDING PROBLEM *cont.*

- Sxx Plot



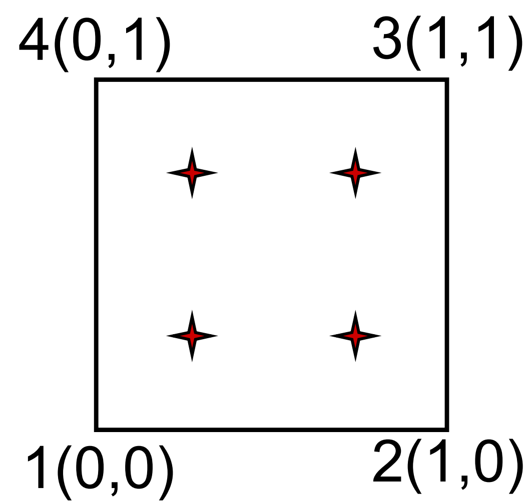
- Stress is constant along the x-axis (pure bending)
- linear through the height of the beam
- Deflection is much higher than CST element. In fact, CST element is too stiff. However, stress is inaccurate.

BEAM BENDING PROBLEM *cont.*

- Caution:
  - In numerical integration, we did not calculate stress at node points. Instead, we calculate stress at integration points.
  - Let's calculate stress at the bottom surface for element 1 in the beam bending problem.
  - Nodal Coordinates:1(0,0), 2(1,0), 3(1,1), 4(0,1)
  - Nodal Displacements:

    u = [0, 0.0002022, -0.0002022, 0]

    v = [0, 0.0002022, 0.0002022, 0]



  - Shape functions and derivatives

$$N_1 = (x-1)(y-1) \qquad \partial N_1/\partial x = (y-1) \qquad \partial N_1/\partial y = (x-1)$$

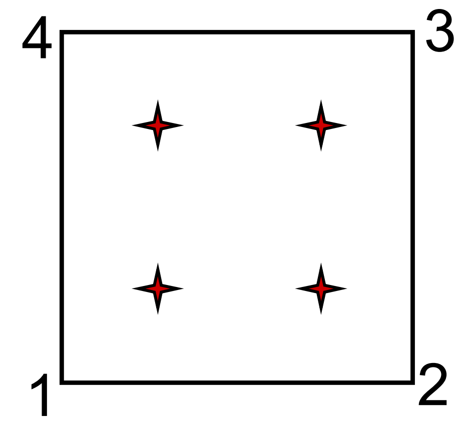
$$N_2 = -x(y-1) \qquad \partial N_2/\partial x = -(y-1) \qquad \partial N_2/\partial y = -x$$

$$N_3 = xy \qquad \partial N_3/\partial x = y \qquad \partial N_3/\partial y = x$$

$$N_4 = -(x-1)y \qquad \partial N_4/\partial x = -y \qquad \partial N_4/\partial y = -(x-1)$$

BEAM BENDING PROBLEM *cont.*

- At bottom surface, y = 0



$$\partial N_1/\partial x = -1 \qquad \partial N_1/\partial y = x-1$$
$$\partial N_2/\partial x = 1 \qquad \partial N_2/\partial y = -x$$
$$\partial N_3/\partial x = 0 \qquad \partial N_3/\partial y = x$$
$$\partial N_4/\partial x = 0 \qquad \partial N_4/\partial y = -(x-1)$$

u = [0, 0.0002022, -0.0002022, 0]

v = [0, 0.0002022, 0.0002022, 0]

- Strain

$$\varepsilon_{xx} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} u_I = 1 \times 0.0002022$$

$$\varepsilon_{yy} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} v_I = -0.0002022 \times x + 0.0002022 \times x = 0$$

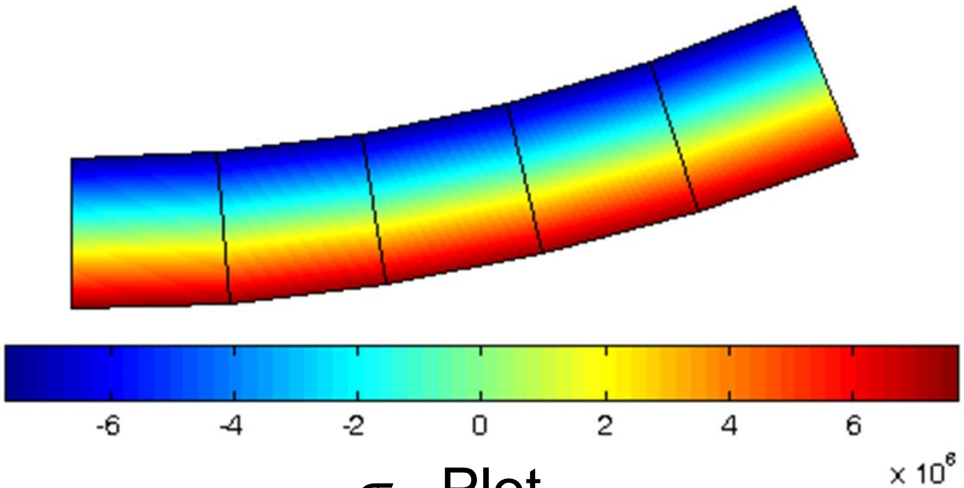
$$\gamma_{xy} = \sum_{I=1}^{4} \left(\frac{\partial N_I}{\partial x} v_I + \frac{\partial N_I}{\partial y} u_I \right) = 0.0002022 - 0.0004044x$$

- Stress:

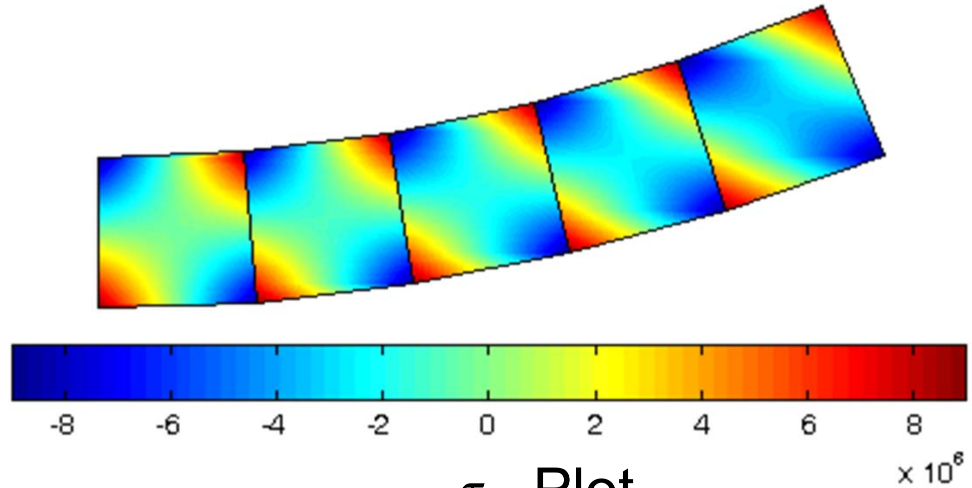
$$\{\sigma\} = [\mathbf{C}]\{\varepsilon\} = \{4.44,\ 1.33,\ 1.55\} \times 10^7$$

RECTANGULAR ELEMENT

- y-normal stress and shear stress are supposed to be zero.



σ_{yy} Plot



τ_{xy} Plot

ε_{xx} is a linear function of y alone

ε_{yy} is a linear function of x alone

γ_{xy} is a linear function of x and y

$$\varepsilon_{xx} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} u_I \qquad \varepsilon_{yy} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} v_I$$

$$\partial N_1 / \partial x = (y-1) \qquad \partial N_1 / \partial y = (x-1)$$

$$\partial N_2 / \partial x = -(y-1) \qquad \partial N_2 / \partial y = -x$$

$$\partial N_3 / \partial x = y \qquad \partial N_3 / \partial y = x$$

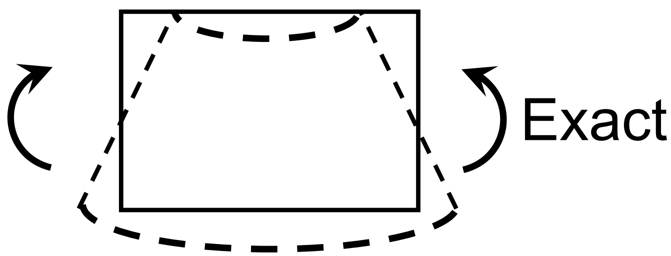
$$\partial N_4 / \partial x = -y \qquad \partial N_4 / \partial y = -(x-1)$$

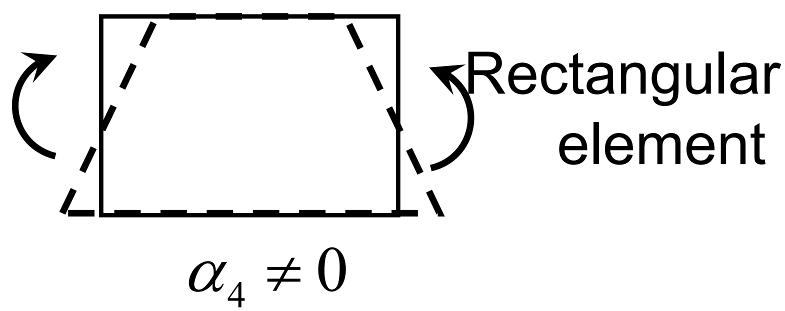
Rectangular Element in Bending

- Discussions
  - Can't represent constant shear force problem because ε_{xx} must be a linear function of *x*.
  - Even if ε_{xx} can represent linear strain in y-direction, the rectangular element can't represent pure bending problem accurately.
  - Spurious shear strain makes the element too stiff.

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$
$$v = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$

$$\varepsilon_{xx} = \alpha_2 + \alpha_4 y$$
$$\varepsilon_{yy} = \beta_3 + \beta_4 x$$
$$\gamma_{xy} = (\alpha_3 + \beta_2) + \alpha_4 x + \beta_4 y$$



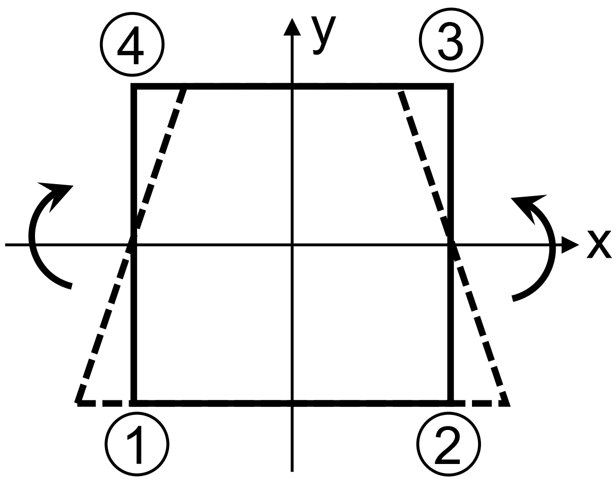


Rectangular Element in Bending

- When $-u_1 = u_2 = -u_3 = u_4 = \alpha_4$

$$u(x,y) = \sum_{i=1}^{4} N_i(x,y)u_i = -\alpha_4 xy$$

$$v(x,y) = \sum_{i=1}^{4} N_i(x,y)v_i = 0$$



- Strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -\alpha_4 y$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\alpha_4 x$$

$\Rightarrow$

Stress

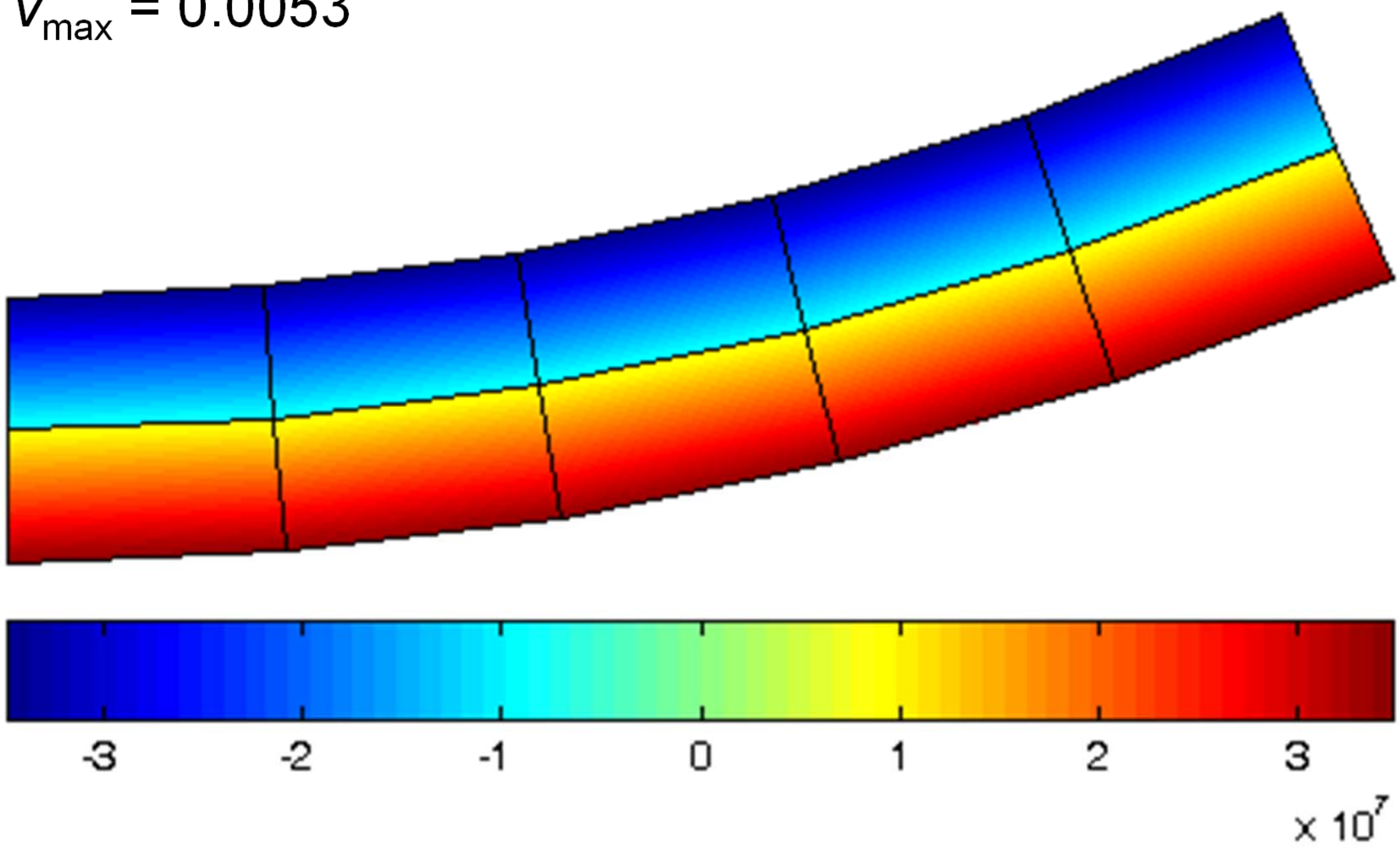
$$\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx} = -\frac{\alpha_4 E}{(1-\nu^2)}y$$

$$\sigma_{yy} = \frac{\nu E}{1-\nu^2}\varepsilon_{xx} = -\frac{\nu E \alpha_4}{(1-\nu^2)}y$$

$$\tau_{xy} = G\gamma_{xy} = -\frac{E\alpha_4}{2(1+\nu)}x$$

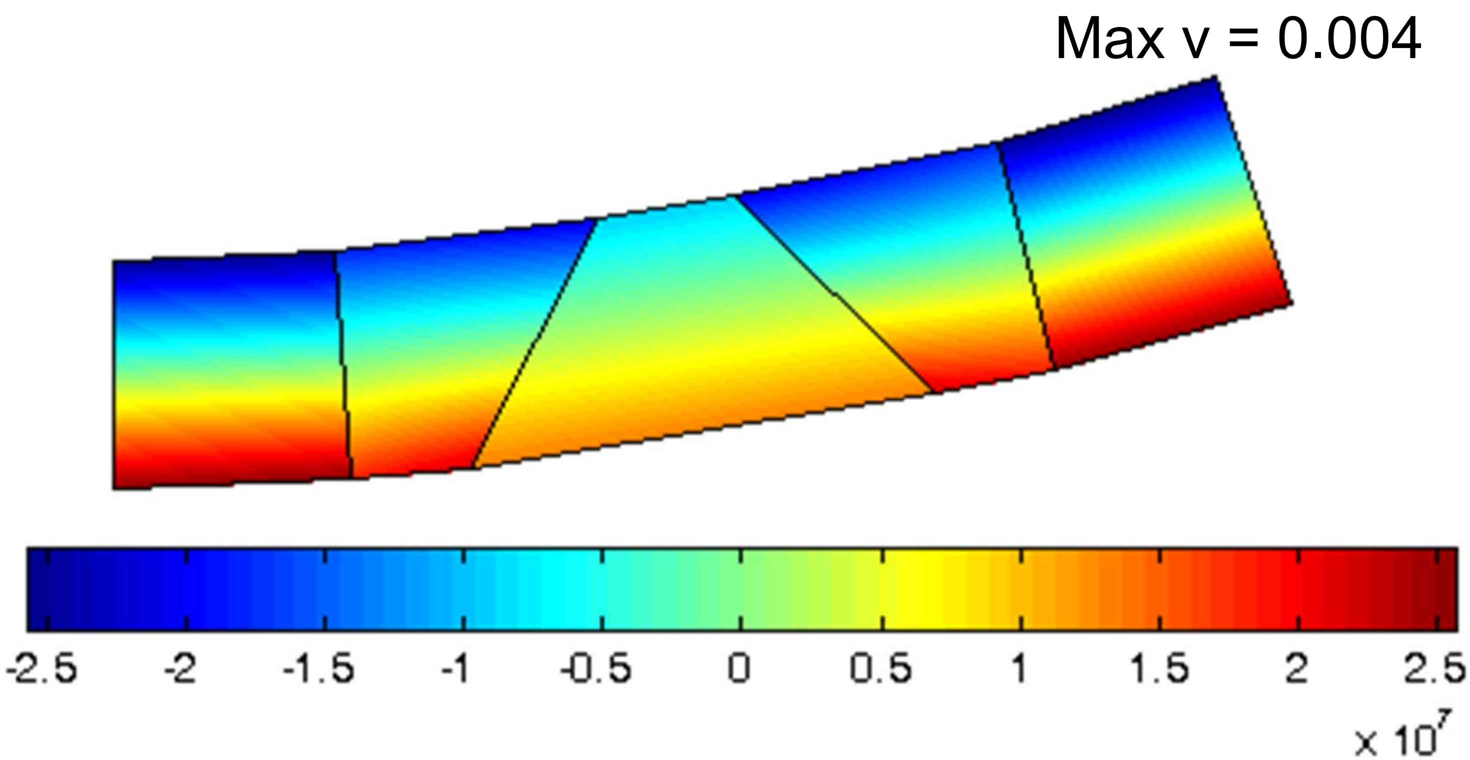
RECTANGULAR ELEMENT

- Two-Layer Model
  - $\sigma_{xx} = 3.48\times10^7$
  - $v_{max} = 0.0053$



BEAM BENDING PROBLEM *cont.*

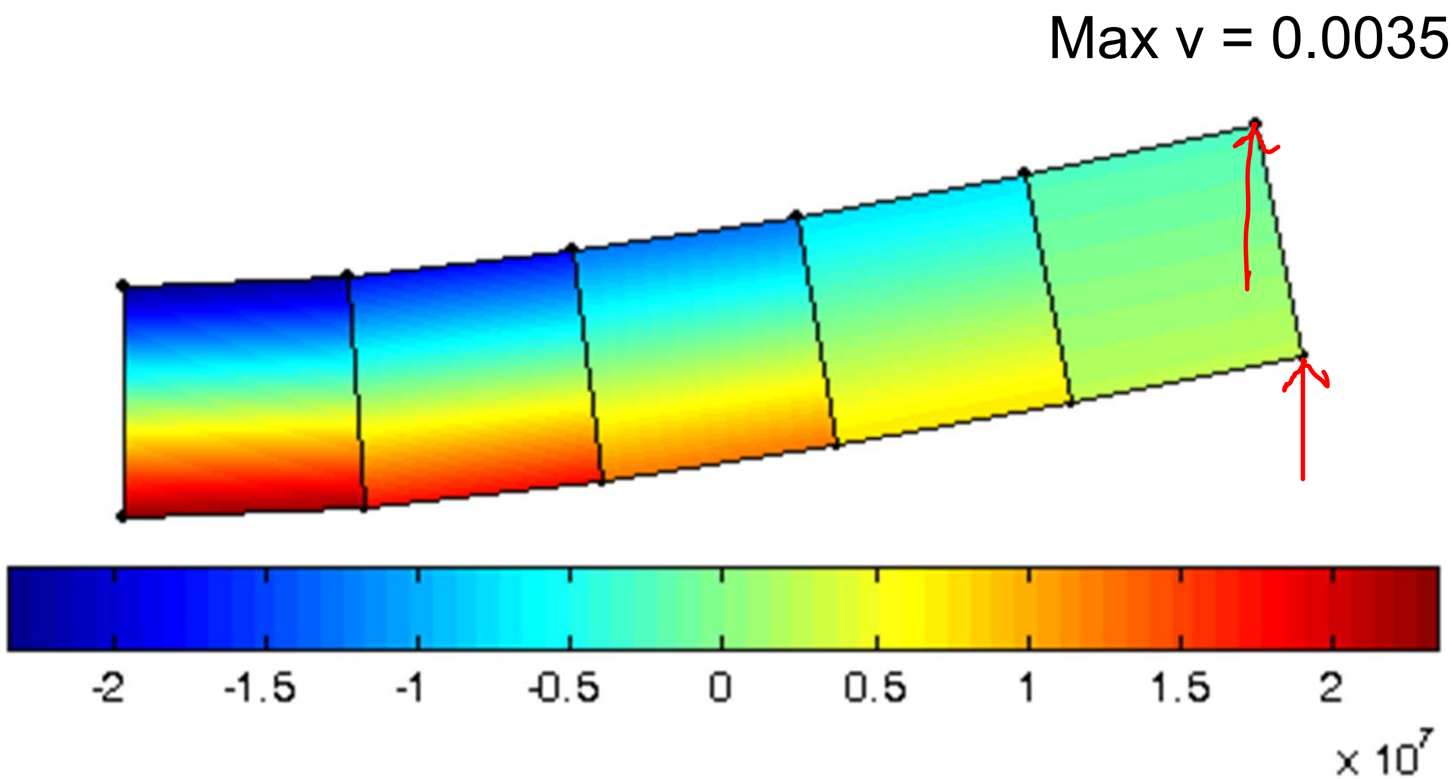
- Distorted Element



- As element is distorted, the solution is not accurate any more.

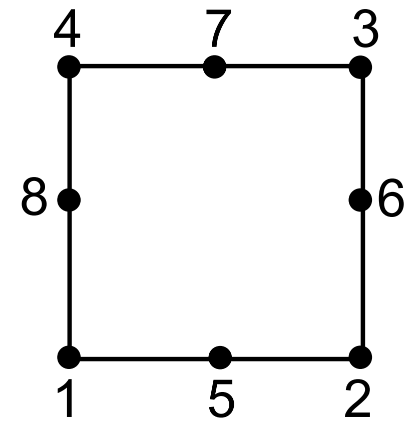
BEAM BENDING PROBLEM *cont.*

- Constant Shear Force Problem



- Sxx is supposed to change linearly along x-axis. But, the element is unable to represent linear change of stress along x-axis. Why?
- Exact solution: v = 0.005 m and σ_{xx} = 6e7 Pa.

BEAM BENDING PROBLEM *cont.*



- Higher-Order Element?
  - 8-Node Rectangular Element

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy$$
$$+ a_5 y^2 + a_6 x^2 y + a_7 xy^2$$

  - Strain

$$\frac{\partial u(x,y)}{\partial x} = a_1 + 2a_3 x + a_4 y + 2a_6 xy + a_7 y^2$$

  - Can this element accurately represent pure bending and constant shear force problem?

1

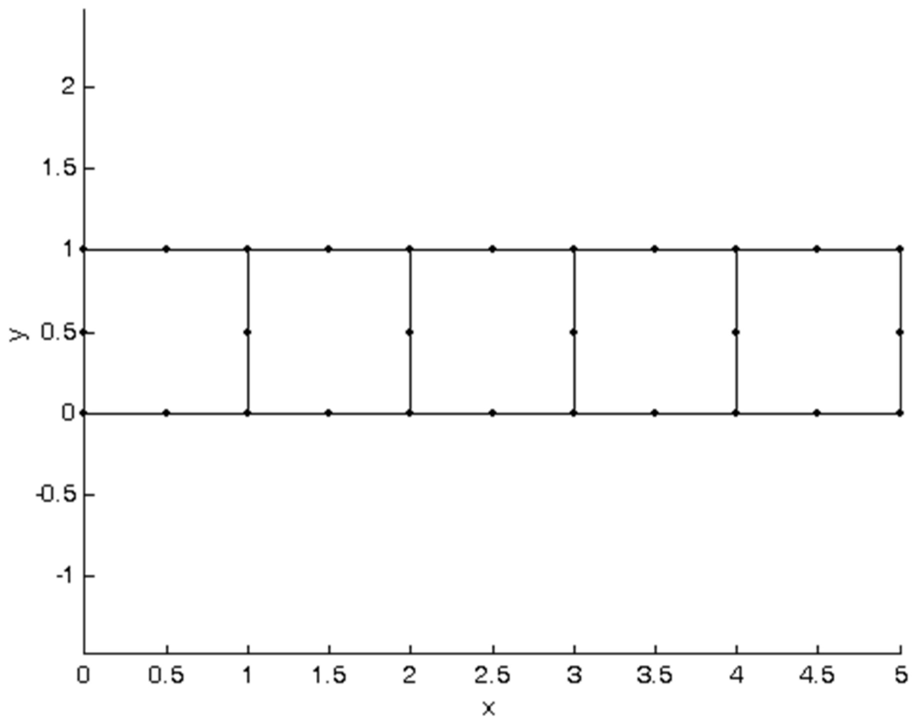
x y

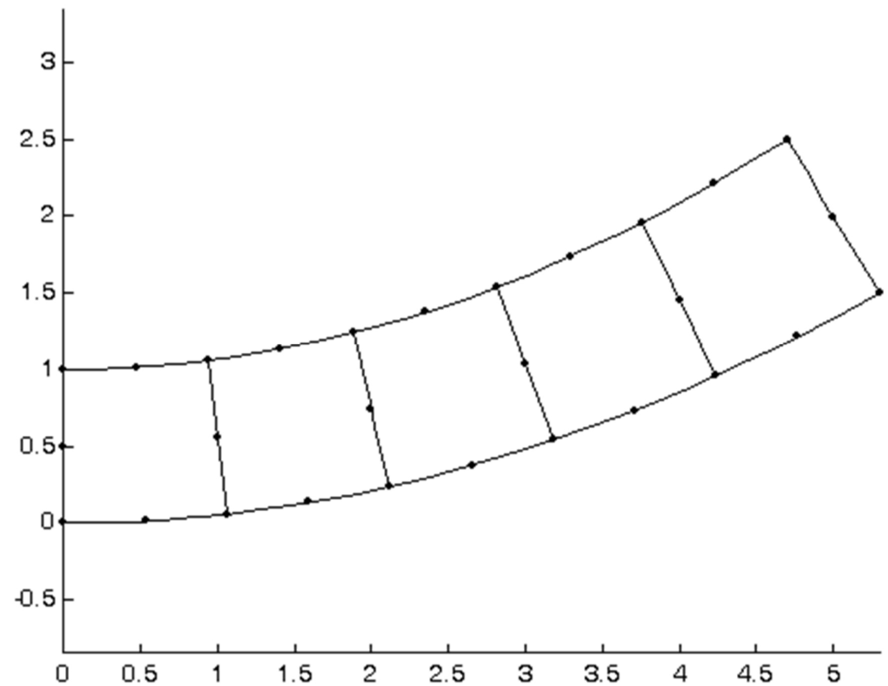
x^2 xy y^2

x^3 x^2y xy^2 y^3

BEAM BENDING PROBLEM *cont.*

- 8-Node Rectangular Elements

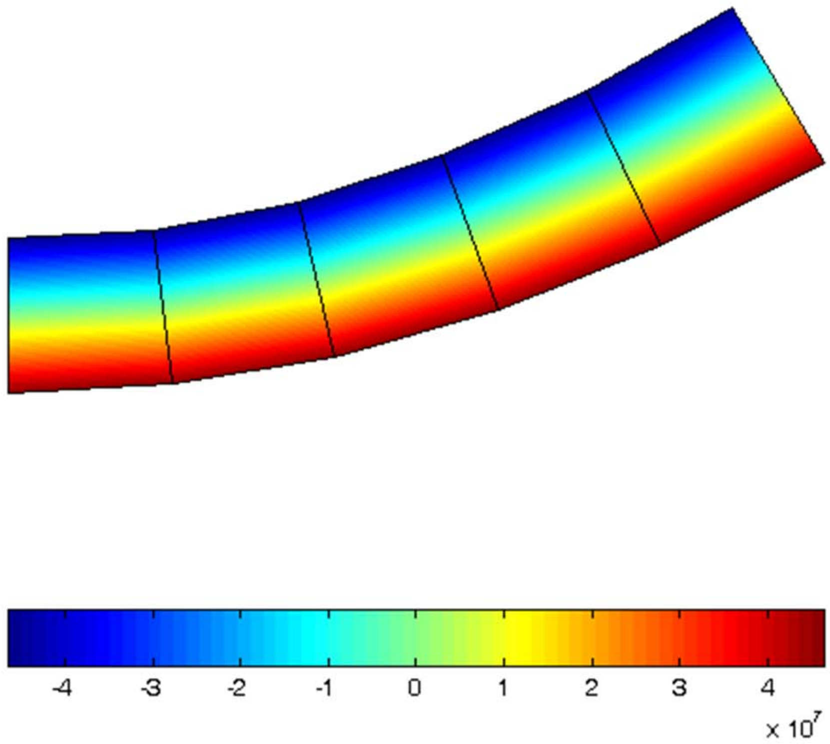




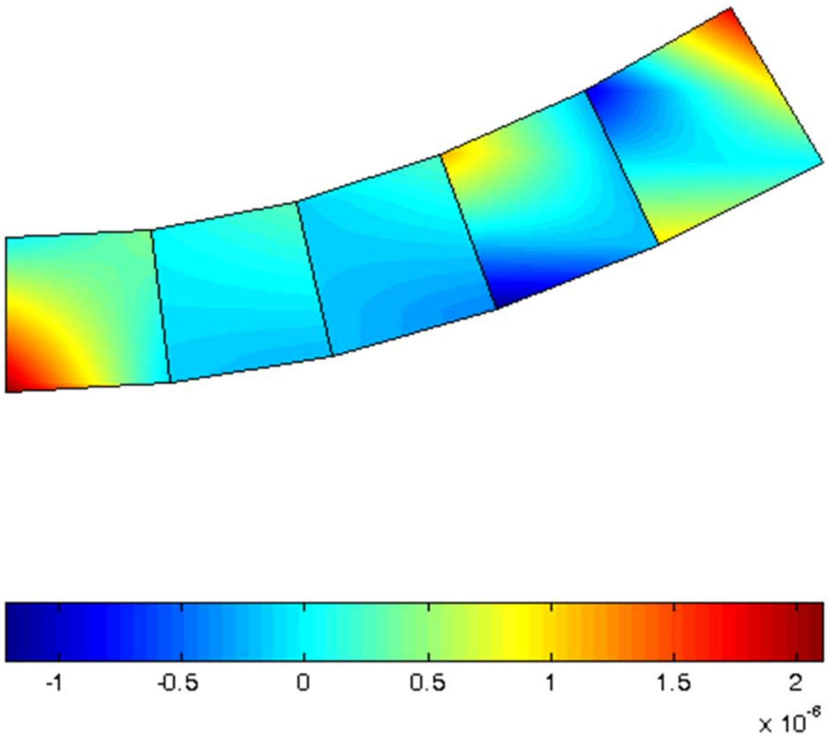
- Tip Displacement = 0.0075 m,   Exact!

BEAM BENDING PROBLEM *cont.*

- If the stress at the bottom surface is calculated, it will be the exact stress value.



Sxx



Syy