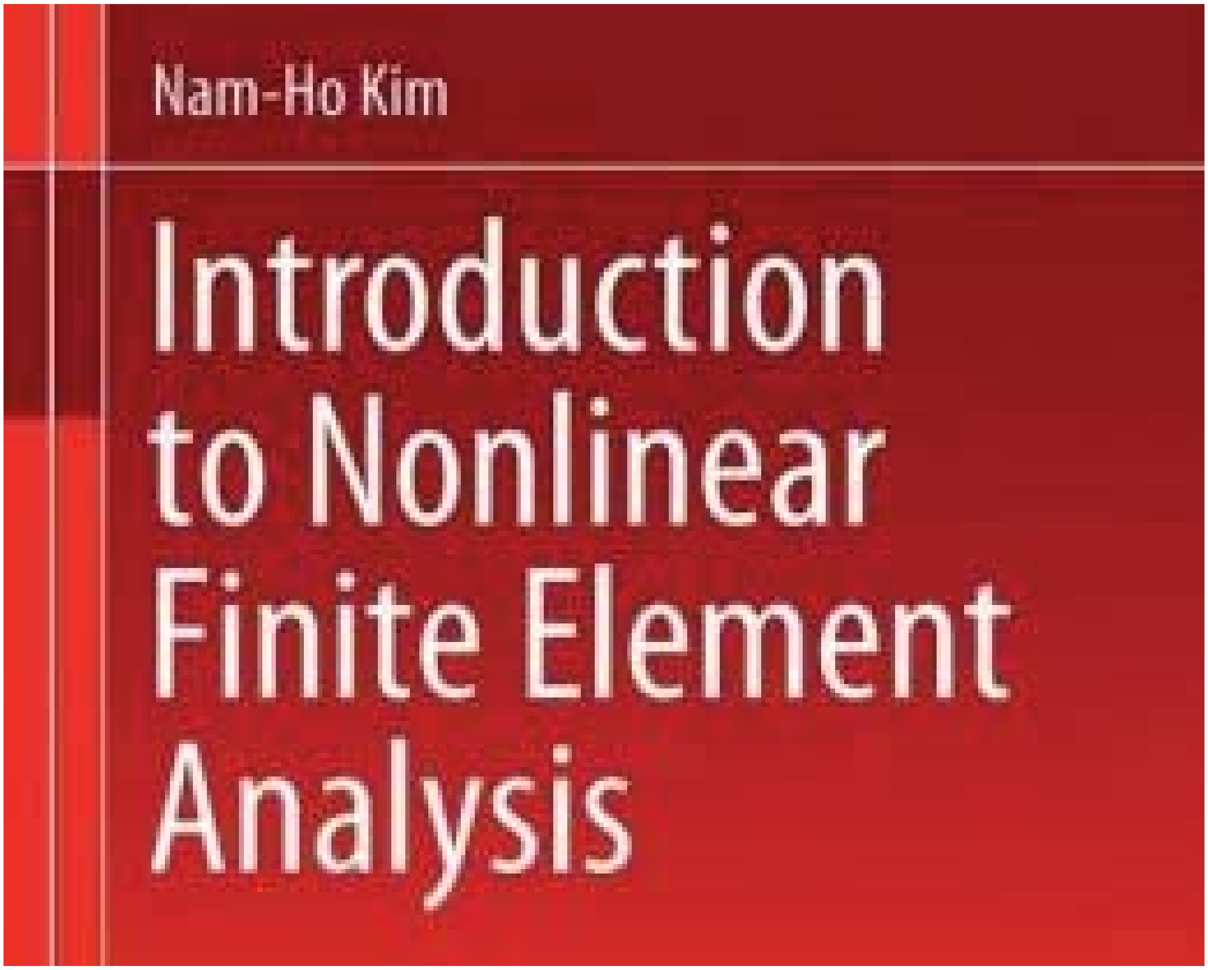




**CHAP 1**

**Preliminary Concepts and Linear Finite Elements**

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1.1

# INTRODUCTION

## Background

- Finite Element Method (FEM):
  - a powerful tool for solving partial differential equations and integro-differential equations
- Linear FEM:
  - methods of modeling and solution procedure are well established
- Nonlinear FEM:
  - different modeling and solution procedures based on the characteristics of the problems → complicated
  - many textbooks in the nonlinear FEMs emphasize complicated theoretical parts or advanced topics
- This book:
  - to simply introduce the nonlinear finite element analysis procedure and to clearly explain the solution procedure
  - detailed theories, solution procedures, and implementation using MATLAB for only representative problems



## Chapter Outline

2. Vector and Tensor Calculus
   - Preliminary to understand mathematical derivations in other chapters
3. Stress and Strain
   - Review of mechanics of materials and elasticity
4. Mechanics of Continuous Bodies
   - Energy principles for structural equilibrium (principle of minimum potential energy)
   - Principle of virtual work for more general non-potential problems
5. Finite Element Method
   - Discretization of continuum equations and approximation of solution using piecewise polynomials
   - Introduction to MATLAB program ELAST3D

1.2

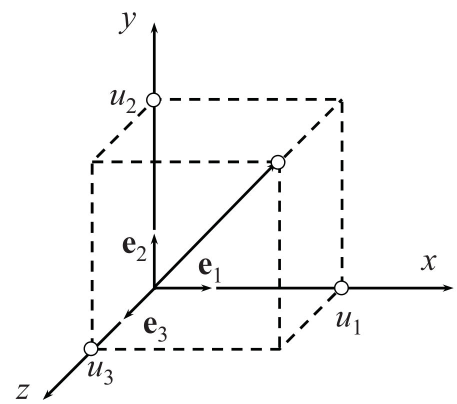
# VECTOR AND TENSOR CALCULUS



## Vector and Tensor

- Vector: Collection of scalars
- Cartesian vector: Euclidean vector defined using Cartesian coordinates
  - 2D, 3D Cartesian vectors

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \text{ or } \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$




  - Using basis vectors: $\mathbf{e}_1 = \{1, 0, 0\}^T$, $\mathbf{e}_2 = \{0, 1, 0\}^T$, $\mathbf{e}_3 = \{0, 0, 1\}^T$

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$$

# Index Notation and Summation Rule

- Index notation: Any vector or matrix can be expressed in terms of its indices

$$\mathbf{v} = [v_i] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \mathbf{A} = [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

- **Einstein summation convention**

$$\sum_{k=1}^{3} a_k b_k = a_k b_k$$

Repeated indices mean summation!!

  - In this case, k is a dummy variable (can be j or i) $\quad a_k b_k = a_j b_j$
  - **The same index cannot appear more than twice**

- Basis representation of a vector
  - Let $\mathbf{e}_k$ be the basis of vector space V
  - Then, any vector in V can be represented by $\mathbf{w} = \sum_{k=1}^{N} w_k \mathbf{e}_k = w_k \mathbf{e}_k$



# Index Notation and Summation Rule cont.

- Examples
  - Matrix multiplication: $\mathbf{C} = \mathbf{A} \cdot \mathbf{B} \quad C_{ij} = A_{ik} B_{kj}$
  - **Trace** operator: $\mathrm{tr}(\mathbf{A}) = A_{11} + A_{22} + A_{33} = A_{kk}$
  - Dot product: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = u_k v_k$
  - Cross product: $\mathbf{u} \times \mathbf{v} = u_j v_k (\mathbf{e}_j \times \mathbf{e}_k) = e_{ijk} u_j v_k \mathbf{e}_i$

**Permutation symbol**

$$e_{ijk} = \begin{cases} 0 & \text{unless } i, j, k \text{ are distinct} \\ +1 & \text{if } (i, j, k) \text{ is an even permutation} \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation} \end{cases}$$

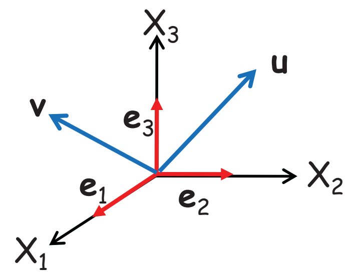
  - **Contraction**: double dot product

$$J = \mathbf{A} : \mathbf{B} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} B_{ij} = A_{ij} B_{ij}$$

# Cartesian Vector

- Cartesian Vectors

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = u_i\mathbf{e}_i$$
$$\mathbf{v} = v_j\mathbf{e}_j$$




- Dot product

$$\mathbf{u}\cdot\mathbf{v} = (u_i\mathbf{e}_i)\cdot(v_j\mathbf{e}_j) = u_i v_j(\mathbf{e}_i\cdot\mathbf{e}_j) = u_i v_j\delta_{ij} = u_i v_i$$

  - **Kronecker delta function**

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

  - **Equivalent to change index j to i, or vice versa**

- How to obtain Cartesian components of a vector

$$\mathbf{e}_i\cdot\mathbf{v} = \mathbf{e}_i\cdot(v_j\mathbf{e}_j) = v_j\delta_{ij} = v_i$$ **Projection**

- Magnitude of a vector (norm): $\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$



# Notation Used Here

| Direct tensor notation | Tensor component notation | Matrix notation |
|---|---|---|
| $\alpha = \mathbf{a}\cdot\mathbf{b}$ | $\alpha = a_i b_i$ | $\alpha = \mathbf{a}^T\mathbf{b}$ |
| $\mathbf{A} = \mathbf{a}\otimes\mathbf{b}$ | $A_{ij} = a_i b_j$ | $\mathbf{A} = \mathbf{a}\mathbf{b}^T$ |
| $\mathbf{b} = \mathbf{A}\cdot\mathbf{a}$ | $b_i = A_{ij}a_j$ | $\mathbf{b} = \mathbf{A}\mathbf{a}$ |
| $\mathbf{b} = \mathbf{a}\cdot\mathbf{A}$ | $b_j = a_i A_{ij}$ | $\mathbf{b}^T = \mathbf{a}^T\mathbf{A}$ |

# Tensor and Rank

- Tensor
  - A tensor is an extension of scalar, vector, and matrix (multidimensional array in a given basis)
  - **A tensor is independent of any chosen frame of reference**
  - Tensor field: a tensor-valued function associated with each point in geometric space
- Rank of Tensor
  - No. of indices required to write down the components of tensor
  - Scalar (rank 0), vector (rank 1), matrix (rank 2), etc
  - Every tensor can be expressed as a linear combination of rank 1 tensors
  - Rank 1 tensor $\mathbf{v}$: $v_i$
  - Rank 2 tensor $\mathbf{A}$: $A_{ij}$
  - Rank 4 tensor $\mathbf{C}$: $C_{ijkl}$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Rank-2 stress tensor



# Tensor Operations

- Basic rules for tensors

$$(\mathbf{TS})\mathbf{R} = \mathbf{T}(\mathbf{SR})$$
$$\mathbf{T}(\mathbf{S}+\mathbf{R}) = \mathbf{TS} + \mathbf{TR}$$
$$\alpha(\mathbf{TS}) = (\alpha\mathbf{T})\mathbf{S} = \mathbf{T}(\alpha\mathbf{S})$$
$$\mathbf{1T} = \mathbf{T1} = \mathbf{T}$$

Different notations

$$\mathbf{TS} = \mathbf{T}\cdot\mathbf{S}$$

**Identity tensor**

$$\mathbf{1} = [\delta_{ij}]$$

- **Tensor (dyadic) product**: increase rank

$$\mathbf{A} = \mathbf{u}\otimes\mathbf{v} = u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j \qquad A_{ij} = u_i v_j \qquad \mathbf{A}^T = A_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$
$$(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{w} = \mathbf{u}(\mathbf{v}\cdot\mathbf{w})$$
$$\mathbf{w}\cdot(\mathbf{u}\otimes\mathbf{v}) = \mathbf{v}(\mathbf{w}\cdot\mathbf{u})$$
$$(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x}) = (\mathbf{v}\cdot\mathbf{w})\mathbf{u}\otimes\mathbf{x} \qquad \mathbf{u}\otimes\mathbf{v} \neq \mathbf{v}\otimes\mathbf{u}$$

- Rank-4 tensor: $\mathbf{D} = D_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$

# Tensor Operations cont.

- Symmetric and skew tensors
  - Symmetric $\mathbf{S} = \mathbf{S}^T$
  - Skew $\mathbf{W} = -\mathbf{W}^T$
  - **Every tensor can be uniquely decomposed by symmetric and skew tensors**

$$\boxed{\mathbf{T} = \mathbf{S} + \mathbf{W}}$$

$$\mathbf{S} = \tfrac{1}{2}(\mathbf{T} + \mathbf{T}^T)$$

$$\mathbf{W} = \tfrac{1}{2}(\mathbf{T} - \mathbf{T}^T)$$

  - Note: **W** has zero diagonal components and $W_{ij} = -W_{ji}$
- Properties – Let **A** be a symmetric tensor

$$\boxed{\begin{aligned}\mathbf{A} : \mathbf{W} &= 0\\ \mathbf{A} : \mathbf{T} &= \mathbf{A} : \mathbf{S}\end{aligned}}$$



# Example

- Displacement gradient can be considered a tensor (rank 2)

$$\nabla\mathbf{u} = \left[\frac{\partial \mathbf{u}}{\partial \mathbf{X}}\right] = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix}$$

$$\mathrm{sym}(\nabla\mathbf{u}) = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) & \frac{\partial u_2}{\partial X_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) & \frac{\partial u_3}{\partial X_3} \end{bmatrix}$$

Strain tensor

$$\mathrm{skew}(\nabla\mathbf{u}) = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} - \frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} - \frac{\partial u_3}{\partial X_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} - \frac{\partial u_2}{\partial X_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} - \frac{\partial u_3}{\partial X_2}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} - \frac{\partial u_3}{\partial X_1}\right) & -\frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} - \frac{\partial u_3}{\partial X_2}\right) & 0 \end{bmatrix}$$

Spin tensor

# Contraction and Trace

- Contraction of rank-2 tensors

$$\mathbf{a} : \mathbf{b} = a_{ij}b_{ij} = a_{11}b_{11} + a_{12}b_{12} + \ldots + a_{32}b_{32} + a_{33}b_{33}$$

  - contraction operator reduces four ranks from the sum of ranks of two tensors

- magnitude (or, norm) of a rank-2 tensor

$$|\mathbf{a}| = \sqrt{\mathbf{a} : \mathbf{a}}$$

- Constitutive relation between stress and strain

$$\sigma = \mathbf{D} : \varepsilon, \quad \sigma_{ij} = D_{ijkl}\varepsilon_{kl}$$

- Trace: part of contraction

$$\mathrm{tr}(A) = A_{ii} = A_{11} + A_{22} + A_{33}$$

  - In tensor notation

$$\mathrm{tr}(\boldsymbol{A}) = \boldsymbol{A} : \mathbf{1} = \mathbf{1} : \boldsymbol{A}$$



# Orthogonal Tensor

- In two different coord.

$$\mathbf{u} = u_i\mathbf{e}_i = u_j^*\mathbf{e}_j^*$$

- Direction cosines

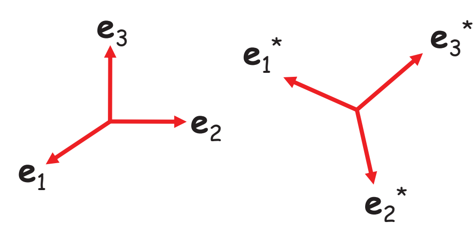
$$\beta = [\beta_{ij}] = \mathbf{e}_i^* \otimes \mathbf{e}_j \qquad \mathbf{e}_i^* = \beta\mathbf{e}_j$$

- Change basis

$$\mathbf{u} = u_j\mathbf{e}_j = u_i^*\mathbf{e}_i^*$$
$$= u_i^*\beta_{ij}\mathbf{e}_j$$

$$u_j = \beta_{ij}u_i^*$$

$$\boxed{\mathbf{u} = \beta^T\mathbf{u}^*}$$




We can also show

$$\mathbf{e}_j = \beta_{ij}\mathbf{e}_i^* \qquad \mathbf{u}^* = \beta\mathbf{u}$$

$$\mathbf{u} = \beta^T\mathbf{u}^* = \beta^T(\beta\mathbf{u}) = (\beta^T\beta)\mathbf{u}$$

$$\beta^{-1} = \beta^T$$

$$\beta^T\beta = \beta\beta^T = \mathbf{1} \qquad \det(\beta) = \pm 1$$

**Orthogonal tensor**

**Rank-2 tensor transformation**

$$\mathbf{T}^* = \beta\mathbf{T}\beta^T, \quad T_{ij}^* = \beta_{ik}T_{kl}\beta_{jl}$$

# Permutation

- The permutation symbol has three indices, but it is not a tensor

$$e_{ijk} = \begin{cases} 1 & \text{if ijk are an even permutation : 123, 231, 312} \\ -1 & \text{if ijk are an odd permutation : 132, 213, 321} \\ 0 & \text{otherwise} \end{cases}$$

- the permutation is zero when any of two indices have the same value: $e_{112} = e_{121} = e_{111} = 0$
- Identity

$$e_{ijk}e_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

- vector product

$$\mathbf{u} \times \mathbf{v} = \mathbf{e}_i e_{ijk} u_j v_k$$



# Dual Vector

- For any skew tensor **W** and a vector **u**

$$\mathbf{u} \cdot \mathbf{W}\mathbf{u} = \mathbf{u} \cdot \mathbf{W}^T\mathbf{u} = -\mathbf{u} \cdot \mathbf{W}\mathbf{u} = 0$$

  - **Wu** and **u** are orthogonal

- Let $W_{ij} = -e_{ijk}w_k$

$$\mathbf{W} = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix} \Rightarrow \mathbf{w} = \begin{Bmatrix} -W_{23} \\ W_{13} \\ -W_{12} \end{Bmatrix}$$

- Then, $W_{ij}u_j = -e_{ijk}w_k u_j = e_{ikj}w_k u_j$

$$\Downarrow$$

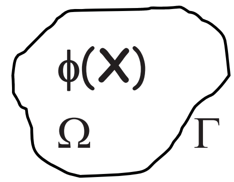
$$\mathbf{W}\mathbf{u} = \mathbf{w} \times \mathbf{u}$$

$\longrightarrow$ Dual vector of skew tensor **W**

$$w_i = -\tfrac{1}{2} e_{ijk} W_{jk}$$

# Vector and Tensor Calculus

- Gradient




$$\boxed{\nabla = \frac{\partial}{\partial \mathbf{X}} = \mathbf{e}_i \frac{\partial}{\partial X_i}}$$

  - Gradient is considered a vector
  - We will often use a simplified notation: $v_{i,j} = \dfrac{\partial v_i}{\partial X_j}$

- Laplace operator

$$\nabla^2 = \nabla \cdot \nabla = \left( \mathbf{e}_i \frac{\partial}{\partial X_i} \right) \cdot \left( \mathbf{e}_j \frac{\partial}{\partial X_j} \right) = \frac{\partial}{\partial X_j} \frac{\partial}{\partial X_j}$$

$$= \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} + \frac{\partial^2}{\partial X_3^2}$$

- Gradient of a scalar field $\phi(\mathbf{X})$: vector

$$\nabla \phi(\mathbf{X}) = \mathbf{e}_i \frac{\partial \phi}{\partial X_i}$$



# Vector and Tensor Calculus

- Gradient of a Tensor Field (increase rank by 1)

$$\nabla \boldsymbol{\phi} = \boldsymbol{\phi} \otimes \nabla = \phi_i \mathbf{e}_i \otimes \mathbf{e}_j \frac{\partial}{\partial X_j} = \frac{\partial \phi_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{e}_j$$

- Divergence (decrease rank by 1)

$$\nabla \cdot \boldsymbol{\phi} = \left( \mathbf{e}_i \frac{\partial}{\partial X_i} \right) \cdot \left( \phi_j \mathbf{e}_j \right) = \frac{\partial \phi_i}{\partial X_i}$$

  - Ex) $\nabla \cdot \boldsymbol{\sigma} = \sigma_{jk,j} \mathbf{e}_k$

- Curl

$$\nabla \times \mathbf{v} = \mathbf{e}_i e_{ijk} v_{k,j}$$

# Integral Theorems

- Divergence Theorem

$$\boxed{\iint_{\Omega} \nabla \cdot \mathbf{A}\, d\Omega = \int_{\Gamma} \mathbf{n} \cdot \mathbf{A}\, d\Gamma}$$

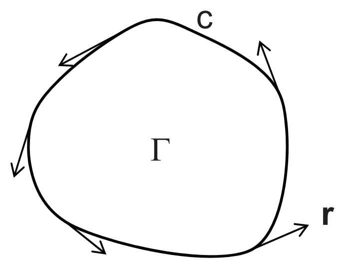
**n**: unit outward normal vector

- Gradient Theorem

$$\iint_{\Omega} \nabla \mathbf{A}\, d\Omega = \int_{\Gamma} \mathbf{n} \otimes \mathbf{A}\, d\Gamma$$

- Stokes Theorem

$$\int_{\Gamma} \mathbf{n} \cdot (\nabla \times \mathbf{v})\, d\Gamma = \oint_{c} \mathbf{r} \cdot \mathbf{v}\, dc$$




- Reynolds Transport Theorem

$$\frac{d}{dt} \iint_{\Omega} \mathbf{A}\, d\Omega = \iint_{\Omega} \frac{\partial \mathbf{A}}{\partial t} d\Omega + \int_{\Gamma} (\mathbf{n} \cdot \mathbf{v}) \mathbf{A}\, d\Gamma$$



# Integration-by-Parts

- u(x) and v(x) are continuously differentiable functions
- 1D

$$\int_a^b u(x) v'(x)\, dx = \left[ u(x) v(x) \right]_a^b - \int_a^b u'(x) v(x)\, dx$$

- 2D, 3D

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v\, d\Omega = \int_{\Gamma} u v n_i\, d\Gamma - \int_{\Omega} u \frac{\partial v}{\partial x_i} d\Omega$$

- For a vector field **v**(x)

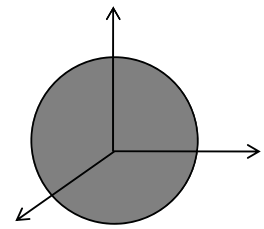
$$\int_{\Omega} \nabla u \cdot \mathbf{v}\, d\Omega = \int_{\Gamma} u(\mathbf{v} \cdot \mathbf{n})\, d\Gamma - \int_{\Omega} u \nabla \cdot \mathbf{v}\, d\Omega$$

- Green's identity

$$\boxed{\int_{\Omega} \nabla u \cdot \nabla v\, d\Omega = \int_{\Gamma} u \nabla v \cdot \mathbf{n}\, d\Gamma - \int_{\Omega} u \nabla^2 v\, d\Omega}$$

# Example: Divergence Theorem

- S: unit sphere ($x^2 + y^2 + z^2 = 1$), $\mathbf{F} = 2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$
- Integrate $\int_S \mathbf{F} \cdot \mathbf{n}\, dS$



$$
\begin{aligned}
\int_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_\Omega \nabla \cdot \mathbf{F}\, d\Omega \\
&= 2\iint_\Omega (1 + y + z)\, d\Omega \\
&= 2\iint_\Omega d\Omega + 2\iint_\Omega y\, d\Omega + 2\iint_\Omega z\, d\Omega \\
&= 2\iint_\Omega d\Omega \\
&= \frac{8\pi}{3}
\end{aligned}
$$



1.3

# STRESS AND STRAIN

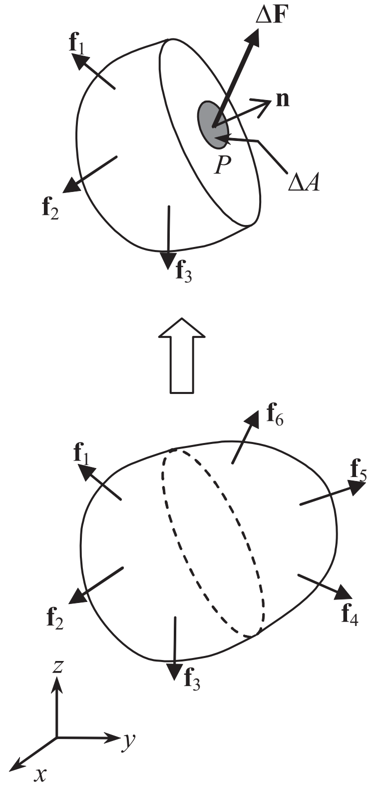
# Surface Traction (Stress)

- Surface traction (Stress)
  - The entire body is in equilibrium with external forces ($\mathbf{f}_1 \sim \mathbf{f}_6$)
  - The imaginary cut body is in equilibrium due to external forces ($\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$) and internal forces
  - Internal force acting at a point P on a plane whose unit normal is $\mathbf{n}$:

$$\mathbf{t}^{(\mathbf{n})} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A}$$

  - The surface traction depends on the unit normal direction $\mathbf{n}$.
  - Surface traction will change as $\mathbf{n}$ changes.
  - unit = force per unit area (pressure)

$$\mathbf{t}^{(\mathbf{n})} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3$$



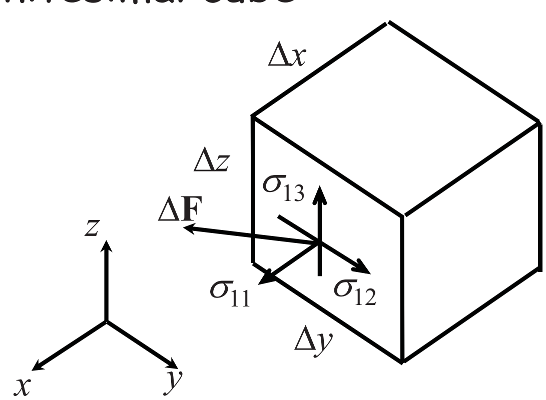



# Cartesian Stress Components

- Surface traction changes according to the direction of the surface.
- Impossible to store stress information for all directions.
- Let's store surface traction parallel to the three coordinate directions.
- Surface traction in other directions can be calculated from them.
- Consider the x-face of an infinitesimal cube

$$\mathbf{t}^{(x)} = t_1^{(x)}\mathbf{e}_1 + t_2^{(x)}\mathbf{e}_2 + t_3^{(x)}\mathbf{e}_3$$

$$\mathbf{t}^{(x)} = \underbrace{\sigma_{11}\mathbf{e}_1}_{\text{Normal stress}} + \underbrace{\sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3}_{\text{Shear stress}}$$



# Stress Tensor

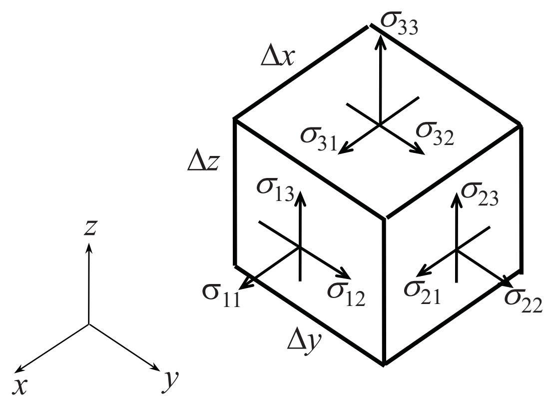
- First index is the face and the second index is its direction
- When two indices are the same, normal stress, otherwise shear stress.
- Continuation for other surfaces.
- Total nine components
- Same stress components are defined for the negative planes.

• Rank-2 Stress Tensor

$$\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

• Sign convention

$$\mathrm{sgn}(\sigma_{11}) = \mathrm{sgn}(\mathbf{n}) \times \mathrm{sgn}(\Delta F_x)$$
$$\mathrm{sgn}(\sigma_{12}) = \mathrm{sgn}(\mathbf{n}) \times \mathrm{sgn}(\Delta F_y)$$

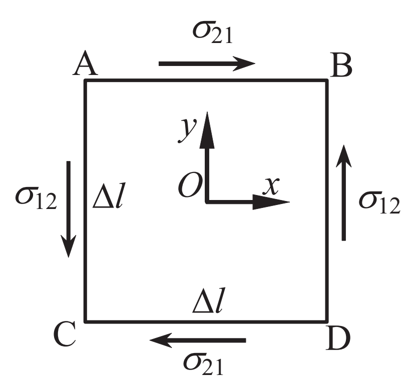





# Symmetry of Stress Tensor

- Stress tensor should be symmetric
  9 components ⟹ 6 components
- Equilibrium of the angular moment

$$\sum M = \Delta l(\sigma_{12} - \sigma_{21}) = 0$$
$$\Rightarrow \sigma_{12} = \sigma_{21}$$




- Similarly for all three directions:

$$\sigma_{12} = \sigma_{21}, \quad \sigma_{23} = \sigma_{32}, \quad \sigma_{13} = \sigma_{31}$$

- Let's use vector notation:

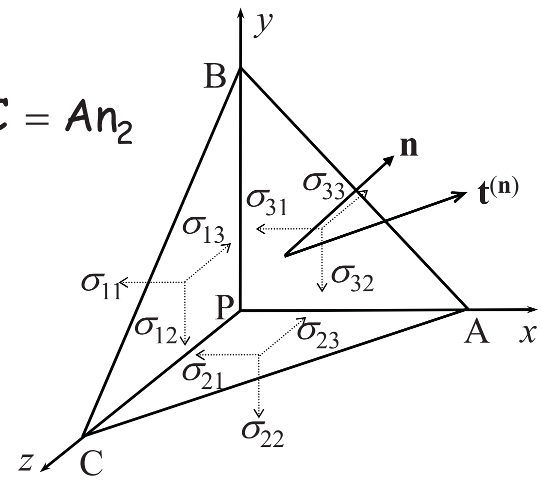
Cartesian components of stress tensor

$$\{\sigma\} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} \qquad [\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

## Stress in Arbitrary Plane

- **If Cartesian stress components are known, it is possible to determine the surface traction acting on any plane.**
- Consider a plane whose normal is **n**.
- Surface area ($\Delta ABC = A$)

$$\Delta PAB = An_3; \ \ \Delta PBC = An_1; \ \ \Delta PAC = An_2$$




- The surface traction

$$\mathbf{t}^{(\mathbf{n})} = t_1^{(\mathbf{n})}\mathbf{e}_1 + t_2^{(\mathbf{n})}\mathbf{e}_2 + t_3^{(\mathbf{n})}\mathbf{e}_3$$

- Force balance

$$\sum F_1 = t_1^{(\mathbf{n})}A - \sigma_{11}An_1 - \sigma_{21}An_2 - \sigma_{31}An_3 = 0$$

$$t_1^{(\mathbf{n})} = \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3$$



## Cauchy's Lemma

- All three-directions

$$t_1^{(\mathbf{n})} = \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3$$
$$t_2^{(\mathbf{n})} = \sigma_{12}n_1 + \sigma_{22}n_2 + \sigma_{32}n_3$$
$$t_3^{(\mathbf{n})} = \sigma_{13}n_1 + \sigma_{23}n_2 + \sigma_{33}n_3$$

- Tensor notation

$$\boxed{\mathbf{t}^{(\mathbf{n})} = \mathbf{n}\cdot\sigma \quad \Rightarrow \quad \mathbf{t}^{(\mathbf{n})} = \sigma\cdot\mathbf{n}}$$

  - stress tensor; completely characterize the state of stress at a point
- Cauchy's Lemma
  - the surface tractions acting on opposite sides of the same surface are equal in magnitude and opposite in direction

$$\mathbf{t}^{(\mathbf{n})} = -\mathbf{t}^{(-\mathbf{n})}$$

# Projected Stresses

- Normal stress $\sigma(\mathbf{n}) = \mathbf{t}^n \cdot \mathbf{n} = \mathbf{n} \cdot \sigma \cdot \mathbf{n} = \sigma_{ij} n_i n_j$
- Shear stress $\tau(\mathbf{n}) = \sqrt{\left\| \mathbf{t}^n \right\|^2 - \sigma^2(\mathbf{n})}, \quad \tau(\mathbf{n}) = \mathbf{t}^n - \mathbf{n}\sigma(\mathbf{n})$
- Principal stresses $\mathbf{t}^n \parallel \mathbf{n} \quad \Rightarrow \quad \sigma_1, \sigma_2, \sigma_3$
- Mean stress (hydrostatic pressure)

$$p = \sigma_m = \frac{1}{3}\mathrm{tr}(\sigma) = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

- **Stress deviator**

$$\mathbf{s} = \sigma - \sigma_m \mathbf{1} = \mathbf{I}_{dev} : \sigma$$

$$\mathbf{s} = \begin{bmatrix} \sigma_{11} - \sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{11} - \sigma_m & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{11} - \sigma_m \end{bmatrix}$$

$$I_{ijkl} = \tfrac{1}{3}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Rank-4 identity tensor

$$\mathbf{I}_{dev} = \mathbf{I} - \tfrac{1}{3}\mathbf{1} \otimes \mathbf{1}$$

Rank-4 deviatoric identity tensor

$$\mathbf{I}_{dev} : \mathbf{1} = \mathbf{0}, \quad \mathbf{I}_{dev} : \mathbf{s} = \mathbf{s}$$



# Principal Stresses

- Normal & shear stress change as **n** changes
  - Is there a plane on which the normal (or shear)stress becomes the maximum?
- There are at least **three mutually perpendicular planes** on which the normal stress attains an extremum
  - Shear stresses are zero on these planes → Principal directions
  - Traction $\mathbf{t}^{(n)}$ is parallel to surface normal **n**

$$\mathbf{t}^{(n)} = \sigma_n \mathbf{n} \quad \Longrightarrow \quad \sigma \cdot \mathbf{n} = \underbrace{\sigma_n}_{\text{Principal stress}} \underbrace{\mathbf{n}}_{\text{Principal direction}}$$

- Eigenvalue problem

$$[\sigma - \sigma_n \mathbf{1}] \cdot \mathbf{n} = \mathbf{0}$$

$$\begin{bmatrix} \sigma_{11} - \sigma_n & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_n & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_n \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

# Eigenvalue Problem for Principal Stresses

- The eigenvalue problem has non-trivial solution if and only if the determinant is zero:

$$\begin{vmatrix} \sigma_{11}-\sigma_n & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_n & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_n \end{vmatrix} = 0$$

- The above equation becomes a cubic equation:

$$\sigma_n^3 - I_1\sigma_n^2 + I_2\sigma_n - I_3 = 0$$

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2$$

$$I_3 = |\sigma| = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{13} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2$$

- Three roots are principal stresses

$$\sigma_1 \ge \sigma_2 \ge \sigma_3$$



# Principal Directions

- Stress Invariants: $I_1$, $I_2$, $I_3$
  - independent of the coordinate system
- Principal directions
  - Substitute each principal stress to the eigenvalue problem to get n
  - Since the determinant is zero, an infinite number of solutions exist
  - Among them, choose the one with a unit magnitude

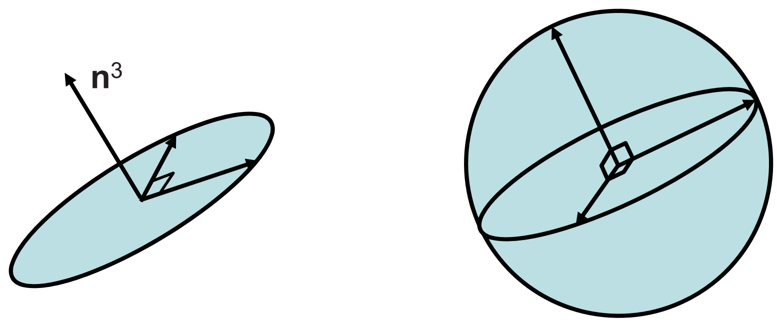
$$\left\|\mathbf{n}^i\right\|^2 = (n_1^i)^2 + (n_2^i)^2 + (n_3^i)^2 = 1, \quad i = 1,2,3$$

- Principal directions are mutually perpendicular

$$\mathbf{n}^i \cdot \mathbf{n}^j = 0, \quad i \neq j$$

# Principal Directions

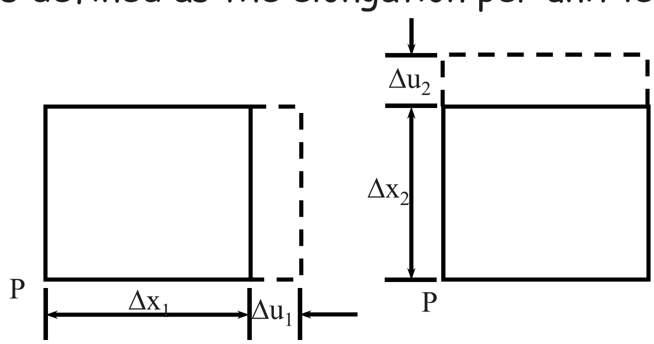
- There are three cases for principal directions:
  1. $\sigma_1$, $\sigma_2$, and $\sigma_3$ are distinct $\Rightarrow$ principal directions are three unique mutually orthogonal unit vectors.
  2. $\sigma_1 = \sigma_2$ and $\sigma_3$ are distinct $\Rightarrow$ $\mathbf{n}^3$ is a unique principal direction, and any two orthogonal directions on the plane that is perpendicular to $\mathbf{n}^3$ are principal directions.
  3. $\sigma_1 = \sigma_2 = \sigma_3$ $\Rightarrow$ any three orthogonal directions are principal directions. This state of stress corresponds to a **hydrostatic pressure**.






# Strains (Simple Version)

- Strain is defined as the elongation per unit length




- Tensile (normal) strains in $x_1$- and $x_2$-directions

$$\varepsilon_{11} = \lim_{\Delta x_1 \to 0} \frac{\Delta u_1}{\Delta x_1} = \frac{\partial u_1}{\partial x_1}$$

$$\varepsilon_{22} = \lim_{\Delta x_2 \to 0} \frac{\Delta u_2}{\Delta x_2} = \frac{\partial u_2}{\partial x_2}$$

Textbook has different, but more rigorous derivations

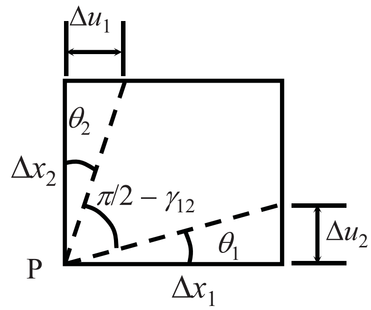
- Strain is a dimensionless quantity. Positive for elongation and negative for compression

# Shear Strain

- Shear strain is the tangent of the change in angle between two originally perpendicular axes

$$\theta_1 \sim \tan\theta_1 = \frac{\Delta u_2}{\Delta x_1}$$

$$\theta_2 \sim \tan\theta_2 = \frac{\Delta u_1}{\Delta x_2}$$




- Shear strain (change of angle)

$$\gamma_{12} = \theta_1 + \theta_2 = \lim_{\Delta x_1 \to 0} \frac{\Delta u_2}{\Delta x_1} + \lim_{\Delta x_2 \to 0} \frac{\Delta u_1}{\Delta x_2} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}$$

$$\varepsilon_{12} = \frac{1}{2}\gamma_{12} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)$$

- Positive when the angle between two positive (or two negative) faces is reduced and negative when the angle is increased.
- Valid for small deformation



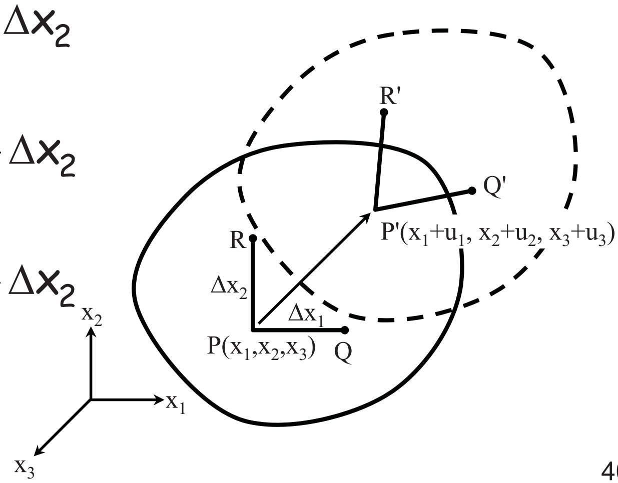
# Strains (Rigorous Version)

- Strain: a measure of deformation
  - Normal strain: change in length of a line segment
  - Shear strain: change in angle between two perpendicular line segments
- Displacement of P = $(u_1, u_2, u_3)$
- Displacement of Q & R

$$u_1^Q = u_1 + \frac{\partial u_1}{\partial x_1}\Delta x_1 \qquad u_1^R = u_1 + \frac{\partial u_1}{\partial x_2}\Delta x_2$$

$$u_2^Q = u_2 + \frac{\partial u_2}{\partial x_1}\Delta x_1 \qquad u_2^R = u_2 + \frac{\partial u_2}{\partial x_2}\Delta x_2$$

$$u_3^Q = u_3 + \frac{\partial u_3}{\partial x_1}\Delta x_1 \qquad u_3^R = u_3 + \frac{\partial u_3}{\partial x_2}\Delta x_2$$



# Displacement Field

- Coordinates of P, Q, and R before and after deformation

$$P:(x_1,x_2,x_3)$$
$$Q:(x_1+\Delta x_1,x_2,x_3)$$
$$R:(x_1,x_1+\Delta x_2,x_3)$$
$$P':(x_1+u_1^P,x_2+u_2^P,x_3+u_3^P)=(x_1+u_1,x_2+u_2,x_3+u_3)$$
$$Q':(x_1+\Delta x_1+u_1^Q,x_2+u_2^Q,x_3+u_3^Q)$$
$$=(x_1+\Delta x_1+u_1+\frac{\partial u_1}{\partial x_1}\Delta x_1,x_2+u_2+\frac{\partial u_2}{\partial x_1}\Delta x_1,x_3+u_3+\frac{\partial u_3}{\partial x_1}\Delta x_1)$$
$$R':(x_1+u_1^R,x_2+\Delta x_2+u_2^R,x_3+u_3^R)$$
$$=(x_1+u_1+\frac{\partial u_1}{\partial x_2}\Delta x_2,x_2+\Delta x_2+u_2+\frac{\partial u_2}{\partial x_2}\Delta x_2,x_3+u_3+\frac{\partial u_3}{\partial x_2}\Delta x_2)$$

- Length of the line segment P'Q'

$$P'Q'=\sqrt{\left(x_1^{P'}-x_1^{Q'}\right)^2+\left(x_2^{P'}-x_2^{Q'}\right)^2+\left(x_3^{P'}-x_3^{Q'}\right)^2}$$



# Deformation Field

- Length of the line segment P'Q'

$$P'Q'=\Delta x_1\sqrt{\left(1+\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_3}{\partial x_1}\right)^2}$$
$$=\Delta x_1\left(1+2\frac{\partial u_1}{\partial x_1}+\left(\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_1}\right)^2+\left(\frac{\partial u_3}{\partial x_1}\right)^2\right)^{1/2}$$
$$\approx\Delta x_1\left(\underbrace{1+\frac{\partial u_1}{\partial x_1}}_{\text{Linear}}+\underbrace{\frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2+\frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}\right)^2+\frac{1}{2}\left(\frac{\partial u_3}{\partial x_1}\right)^2}_{\text{Nonlinear}}\right)\approx\Delta x\left(1+\frac{\partial u_1}{\partial x_1}\right)$$

Nonlinear ⟹ Ignore H.O.T. when displacement gradients are small

- Linear normal strain

$$\boxed{\varepsilon_{11}=\frac{P'Q'-PQ}{PQ}=\frac{\partial u_1}{\partial x_1}\qquad \varepsilon_{22}=\frac{\partial u_2}{\partial x_2},\quad \varepsilon_{33}=\frac{\partial u_3}{\partial x_3}}$$

# Deformation Field

- Shear strain $\gamma_{xy}$
  - change in angle between two lines originally parallel to x– and y–axes

$$\theta_1 = \frac{x_2^{Q'} - x_2^{Q}}{\Delta x_1} = \frac{\partial u_2}{\partial x_1} \qquad \theta_2 = \frac{x_1^{R'} - x_1^{R}}{\Delta x_2} = \frac{\partial u_1}{\partial x_2}$$

$$\gamma_{12} = \theta_1 + \theta_2 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

$$\gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}$$

$$\gamma_{13} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}$$

Engineering shear strain

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)$$

Different notations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

$$\varepsilon = \mathrm{sym}(\nabla \mathbf{u})$$



# Strain Tensor

- Strain Tensor

$$\varepsilon = \varepsilon_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

- Cartesian Components $[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}$

- Vector notation

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix}$$

# Volumetric and Deviatoric Strain

- Volumetric strain (from small strain assumption)

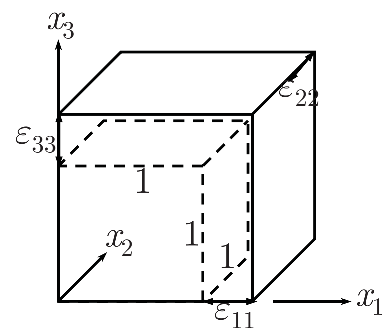
$$\varepsilon_V = \frac{V - V_0}{V_0} = (1+\varepsilon_{11})(1+\varepsilon_{22})(1+\varepsilon_{33}) - 1 \approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$\varepsilon_V = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{kk}$$

- Deviatoric strain

$$\mathbf{e} = \varepsilon - \tfrac{1}{3}\varepsilon_V \mathbf{1} \qquad e_{ij} = \varepsilon_{ij} - \tfrac{1}{3}\varepsilon_V \delta_{ij}$$

$$\mathbf{e} = \mathbf{I}_{dev} : \varepsilon$$

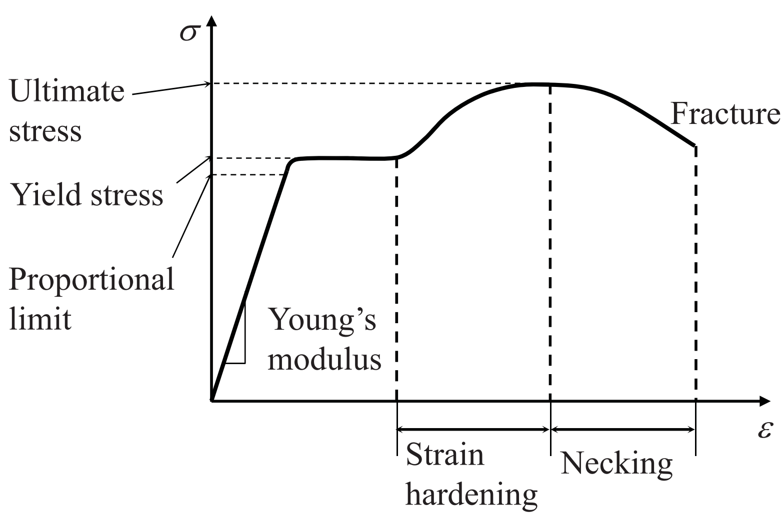



Exercise: Write $\mathbf{I}_{dev}$ in matrix-vector notation



# Stress-Strain Relationship

- Applied Load ⇒ shape change (strain) ⇒ stress
- There must be a relation between stress and strain
- Linear Elasticity: Simplest and most commonly used



# Generalized Hooke's Law

- Linear elastic material $\boxed{\sigma = \mathbf{D} : \varepsilon,}$ $\quad \sigma_{ij} = D_{ijkl}\varepsilon_{kl}$
  - In general, $D_{ijkl}$ has 81 components
  - Due to symmetry in $\sigma_{ij}$, $D_{ijkl} = D_{jikl}$
  - Due to symmetry in $\varepsilon_{kl}$, $D_{ijkl} = D_{ijlk}$
  - from definition of strain energy, $D_{ijkl} = D_{klij}$

  21 independent coeff
- Isotropic material (no directional dependence)
  - Most general 4-th order isotropic tensor

$$\boxed{\begin{aligned} D_{ijkl} &= \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk} \\ &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned}}$$

$$\boxed{\mathbf{D} = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}}$$

  - Have only two independent coefficients (**Lame's constants: $\lambda$ and $\mu$**)



# Generalized Hooke's Law cont.

- Stress-strain relation

$$\boxed{\sigma_{ij} = D_{ijkl}\varepsilon_{kl} = [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]\varepsilon_{kl} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}}$$

  - Volumetric strain: $\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_v$
  - Off-diagonal part: $\sigma_{12} = 2\mu\varepsilon_{12} = \mu\gamma_{12}$ $\quad \mu$ is the shear modulus
  - Bulk modulus K: relation b/w volumetric stress & strain

$$I_1 = 3\sigma_m = \sigma_{jj} = \lambda\varepsilon_{kk}\delta_{jj} + 2\mu\varepsilon_{jj} = (3\lambda + 2\mu)\varepsilon_{kk}$$

$$\boxed{p = \sigma_m = (\lambda + \tfrac{2}{3}\mu)\varepsilon_{kk} = K\varepsilon_v}$$

**Bulk modulus**

  - Substitute $\lambda = K - \frac{2}{3}\mu$ so that we can separate volumetric part
- **Total deform. = volumetric + deviatoric deform.**

# Generalized Hooke's Law cont.

- Stress-strain relation cont.

$$\begin{aligned}\sigma_{ij} &= (K - \tfrac{2}{3}\mu)\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \\ &= K\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - \tfrac{2}{3}\mu\varepsilon_{kk}\delta_{ij} \\ &= K\delta_{ij}\delta_{kl}\varepsilon_{kl} + 2\mu[\delta_{ik}\delta_{jl} - \tfrac{1}{3}\delta_{ij}\delta_{kl}]\varepsilon_{kl} \\ &= \left[K\delta_{ij}\delta_{kl} + 2\mu(\mathbf{I}_{dev})_{ijkl}\right]\varepsilon_{kl}\end{aligned}$$

$$\boxed{\Rightarrow \boldsymbol{\sigma} = \left[\underbrace{K\mathbf{1}\otimes\mathbf{1}}_{\text{Volumetric part}} + \underbrace{2\mu\mathbf{I}_{dev}}_{\text{Deviatoric part}}\right] : \boldsymbol{\varepsilon}}$$

$$\mathbf{e} = \mathbf{I}_{dev} : \boldsymbol{\varepsilon}$$

Deviatoric strain

$$\Rightarrow \boldsymbol{\sigma} = K\varepsilon_v\mathbf{1} + 2\mu\mathbf{e}$$

$$\boldsymbol{\sigma} = \sigma_m\mathbf{1} + \mathbf{s}$$

$$\mathbf{s} = \mathbf{I}_{dev} : \boldsymbol{\sigma}$$

Deviatoric stress

Important for plasticity; plastic deformation only occurs in deviatoric part
volumetric part is always elastic



# Generalized Hooke's Law cont.

- Vector notation
  - The tensor notation is not convenient for computer implementation
  - Thus, we use Voigt notation — $2^{nd}$-order tensor $\Rightarrow$ vector; $4^{th}$-order tensor $\Rightarrow$ matrix
  - Strain (6×1 vector), Stress (6×1 vector), and ***C*** (6×6 matrix)

$$\boldsymbol{\varepsilon} = \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\end{bmatrix} = \begin{bmatrix}u_{1,1}\\ u_{2,2}\\ u_{3,3}\\ u_{1,2}+u_{2,1}\\ u_{2,3}+u_{3,2}\\ u_{1,3}+u_{3,1}\end{bmatrix} \qquad \boldsymbol{\sigma} = \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{13}\end{bmatrix} = \mathbf{D}\boldsymbol{\varepsilon}$$

$\varepsilon_{12} + \varepsilon_{21} = 2\varepsilon_{12}$

You don't need 2 here

# 3D Solid Element cont.

- Elasticity matrix

$$\mathbf{D} = K\mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}_{dev}$$

$$\mathbf{1} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \mathbf{I}_{dev} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

- Relation b/w Lame's constants and Young's modulus and Poisson's ratio

$$\nu = \frac{\lambda}{2(\lambda+\mu)}, \qquad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

$$\mathbf{D} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

Textbook has a definition of D in terms of E and $\nu$



# Plane Stress

- Thin plate-like components parallel to the xy-plane
- The plate is subjected to forces in its in-plane only
- $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$

$$\{\sigma\} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix}$$

- $\varepsilon_{13} = \varepsilon_{23} = 0$, but $\varepsilon_{33} \neq 0$
- $\varepsilon_{33}$ can be calculated from the condition of $\sigma_{33} = 0$:

$$\varepsilon_{33} = -\frac{\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22})$$

## Plane Strain

- Strains with a z subscript are all zero: $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$
- Deformation in the z-direction is constrained, (i.e., $u_3 = 0$)
- can be used if the structure is infinitely long in the z-direction

$$\{\sigma\} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix}$$

- $\sigma_{13} = \sigma_{23} = 0$, but $\sigma_{33} \neq 0$
- $\sigma_{33}$ can be calculated from the condition of $\varepsilon_{33} = 0$:

$$\sigma_{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\varepsilon_{11} + \varepsilon_{22}\right)$$



1.4

# MECHANICS OF CONTINUOUS BODIES

# Governing Equations for Equilibrium

- Governing differential equations for structural equilibrium
  - Three laws of mechanics: conservation of mass, **conservation of linear momentum** and conservation of angular momentum
- Boundary-valued problem: satisfied at every point in $\Omega$
  - Governing D.E. + Boundary conditions
  - Solutions: $C^2$–continuous for truss & solid, $C^4$–continuous for beam
  - Unnecessarily requirements for higher-order continuity
- Energy-based method
  - For conservative system, structural equilibrium when the potential energy has its minimum: **Principle of minimum potential energy**
  - If the solution of BVP exists, then that solution is the minimizing solution of the potential energy
  - When no solution exists in BVP, PMPE may have a natural solution
- Principle of virtual work
  - Equilibrium of the work done by both internal and external forces with small arbitrary virtual displacements



# Balance of Linear Momentum

- Balance of linear momentum

$$\iint_{\Omega} \mathbf{f}^b \, d\Omega + \int_{\Gamma} \mathbf{t}^n \, d\Gamma = \iint_{\Omega} \cancel{\rho \mathbf{a}} \, d\Omega$$

0 for static problem

$\mathbf{f}^b$: body force
$\mathbf{t}^n$: surface traction

- Stress tensor (rank 2):

$$\boldsymbol{\sigma} = \sigma_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \qquad \left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

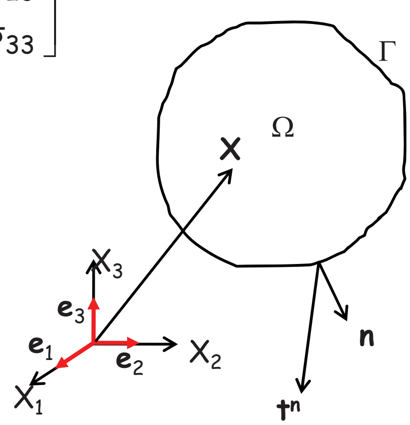
- Surface traction

$$\boxed{\mathbf{t}^n = \mathbf{n} \cdot \boldsymbol{\sigma}}$$

- Cauchy's Lemma

$$\mathbf{t}^n = -\mathbf{t}^{-n}$$

$$\mathbf{t}^n = \mathbf{n} \cdot \boldsymbol{\sigma} \qquad \mathbf{t}^{-n} = -\mathbf{n} \cdot \boldsymbol{\sigma}$$



# Balance of Linear Momentum cont

- Balance of linear momentum

$$\iint_\Omega (\mathbf{f}^b - \rho\mathbf{a})\,d\Omega = -\int_\Gamma \mathbf{n}\cdot\sigma\,d\Gamma = -\iint_\Omega \nabla\cdot\sigma\,d\Omega$$

Divergence Theorem

$$\iint_\Omega [\nabla\cdot\sigma + (\mathbf{f}^b - \rho\mathbf{a})]\,d\Omega = 0$$

$$\Rightarrow \nabla\cdot\sigma + (\mathbf{f}^b - \rho\mathbf{a}) = 0$$

- For a static problem

$$\boxed{\nabla\cdot\sigma + \mathbf{f}^b = 0 \qquad \sigma_{ij,i} + f_j^b = 0}$$

- Balance of angular momentum

$$\iint_\Omega \mathbf{x}\times\mathbf{f}^b\,d\Omega + \int_\Gamma \mathbf{x}\times\mathbf{t}^\mathbf{n}\,d\Gamma = \iint_\Omega \rho\mathbf{x}\times\mathbf{a}\,d\Omega$$

$$\Rightarrow \boxed{\sigma = \sigma^T} \qquad \sigma_{ij} = \sigma_{ji}$$



# Boundary-Valued Problem

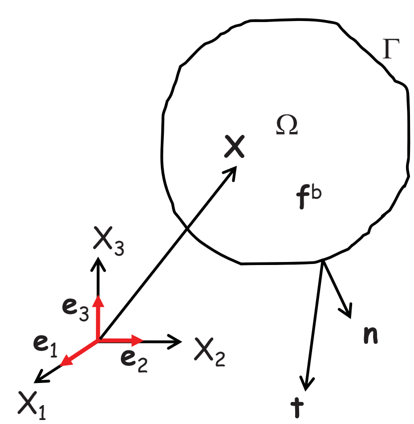
- We want to determine the state of a body in equilibrium
- The equilibrium state (solution) of the body must satisfy
  - local momentum balance equation
  - boundary conditions
- **Strong form** of BVP
  - Given body force $\mathbf{f}^b$, and traction $\mathbf{t}$ on the boundary, find $\mathbf{u}$ such that

$$\nabla\cdot\sigma + \mathbf{f}^b = 0 \qquad (1)$$

and

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma^h \qquad \text{essential BC} \qquad (2)$$

$$\mathbf{t} = \mathbf{n}\cdot\sigma \quad \text{on } \Gamma^s \qquad \text{natural BC} \qquad (3)$$




- Solution space

$$D_A = \left\{\mathbf{u}\in[C^2(\Omega)]^3 \mid \mathbf{u} = 0 \text{ on } \mathbf{x}\in\Gamma^h,\ \sigma\cdot\mathbf{n} = \mathbf{t} \text{ on } \mathbf{x}\in\Gamma^s\right\}$$

# Boundary-Valued Problem cont.

- How to solve BVP
  - To solve the strong form, we want to construct trial solutions that automatically satisfy a part of BVP and find the solution that satisfy remaining conditions.
  - **Statically admissible stress field**: satisfy (1) and (3)
  - **Kinematically admissible displacement field**: satisfy (2) and have piecewise continuous first partial derivative
  - Admissible stress field is difficult to construct. Thus, admissible displacement field is used often



# Principle of Minimum Potential Energy (PMPE)

- Deformable bodies generate internal forces by deformation against externally applied forces
- Equilibrium: balance between internal and external forces
- For elastic materials, the concept of force equilibrium can be extended to energy balance
- Strain energy: stored energy due to deformation (corresponding to internal force)

$$U(\mathbf{u}) \equiv \frac{1}{2}\iint_{\Omega} \sigma(\mathbf{u}) : \varepsilon(\mathbf{u})\, d\Omega \qquad \sigma(\mathbf{u}) = \mathbf{D} : \varepsilon(\mathbf{u})$$

Linear elastic material

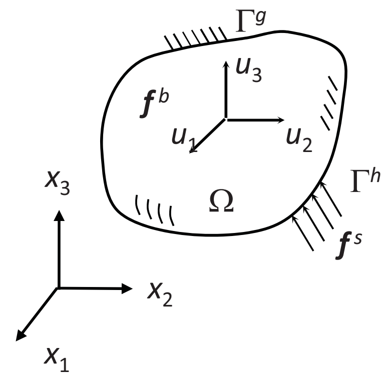
- For elastic material, $U(\mathbf{u})$ is only a function of total displacement $\mathbf{u}$ (independent of path)

# PMPE cont.

- Work done by applied loads (conservative loads)

$$W(\mathbf{u}) = \iint_{\Omega} \mathbf{u} \cdot \mathbf{f}^b \, d\Omega + \int_{\Gamma^s} \mathbf{u} \cdot \mathbf{t} \, d\Gamma.$$

- $U(\mathbf{u})$ is a quadratic function of **u**, while $W(\mathbf{u})$ is a linear function of **u**.




- Potential energy

$$\Pi(\mathbf{u}) = U(\mathbf{u}) - W(\mathbf{u})$$
$$= \frac{1}{2}\iint_{\Omega} \sigma(\mathbf{u}) : \varepsilon(\mathbf{u})\, d\Omega - \iint_{\Omega} \mathbf{u} \cdot \mathbf{f}^b \, d\Omega - \int_{\Gamma^s} \mathbf{u} \cdot \mathbf{t} \, d\Gamma.$$



# PMPE cont.

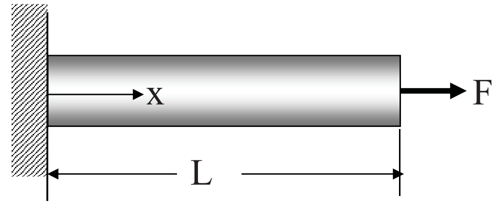
- PMPE: for all displacements that satisfy the boundary conditions, known as kinematically admissible displacements, those which satisfy the boundary-valued problem make the total potential energy stationary on $D_A$
- But, the potential energy is well defined in the space of kinematically admissible displacements

$$\mathbb{Z} = \left\{ \mathbf{u} \in [H^1(\Omega)]^3 \mid \mathbf{u} = 0 \text{ on } \mathbf{x} \in \Gamma^h \right\},$$

$H^1$: first-order derivatives are integrable

- No need to satisfy traction BC (it is a part of potential)
- Less requirement on continuity
- The solution is called a generalized (natural) solution

# Example – Uniaxial Bar




- Strong form

$$\begin{aligned} EAu'' &= 0 \quad & x \in [0, L] \\ u &= 0 \quad & x = 0 \\ EAu'(L) &= F \quad & x = L \end{aligned}$$

- Integrate twice: $EAu(x) = c_1 x + c_2$
- Apply two BCs: $u(x) = \dfrac{Fx}{EA}$ Solution of BVP
- PMPE with assumed solution $u(x) = c_1x + c_2$
- To satisfy KAD space, $u(0) = 0$, → $u(x) = c_1x$
- Potential energy: $U = \dfrac{1}{2}\int_0^L EA(u')^2\,dx = EALc_1^2$

$$W = Fu(L) = FLc_1$$

$$\frac{d\Pi}{dc_1} = \frac{d}{dc_1}(U - W) = EALc_1 - FL = 0 \qquad c_1 = \frac{F}{EA} \quad \Rightarrow \quad u(x) = \frac{Fx}{EA}$$

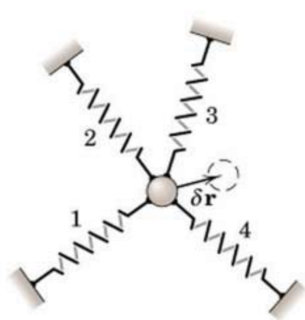


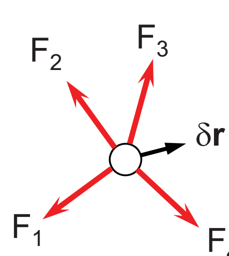
# Virtual Displacement

- Virtual displacement is not experienced but only assumed to exist so that various possible equilibrium positions may be compared to determine the correct one
- Let mass m and springs are in equilibrium at the current position
- Then, a small arbitrary perturbation, $\delta\mathbf{r}$, can be assumed
  - Since $\delta\mathbf{r}$ is so small, the member forces are assumed unchanged
- The work done by virtual displacement is

$$\delta W = \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \mathbf{F}_3 \cdot \delta\mathbf{r} + \mathbf{F}_4 \cdot \delta\mathbf{r} = (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4) \cdot \delta\mathbf{r}$$

- If the current position is in force equilibrium, $\delta W = 0$



# Virtual Displacement Field

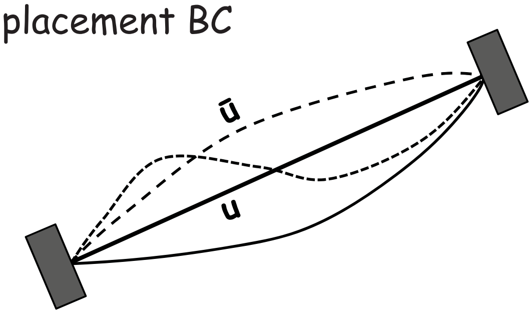
- **Virtual displacement (Space $\mathbb{Z}$)**
  - Small arbitrary perturbation (variation) of real displacement

$$\delta \mathbf{u} = \lim_{\tau \to 0} \frac{1}{\tau}[(\mathbf{u} + \tau\eta) - (\mathbf{u})] = \frac{d}{d\tau}(\mathbf{u} + \tau\eta)\Big|_{\tau=0} = \eta \equiv \bar{\mathbf{u}}.$$

  - Let $\bar{\mathbf{u}}$ be the virtual displacement, then $\mathbf{u} + \bar{\mathbf{u}}$ must be kinematically admissible, too
  - Then, $\bar{\mathbf{u}}$ must satisfy homogeneous displacement BC

$$\mathbf{u} \mapsto \mathbf{u} + \tau\bar{\mathbf{u}} \in \mathbb{V} \quad \Rightarrow \quad \bar{\mathbf{u}} \in \mathbb{Z}$$

$$\mathbb{Z} = \left\{ \bar{\mathbf{u}} \,\middle|\, \bar{\mathbf{u}} \in [H^1(\Omega)]^3,\ \bar{\mathbf{u}}\big|_{\Gamma^h} = 0 \right\}$$




  - Space $\mathbb{Z}$ only includes homogeneous essential BCs

In the literature, $\delta\mathbf{u}$ is often used instead of $\bar{\mathbf{u}}$

- Property of variation

$$\delta\left(\frac{d\mathbf{u}}{dx}\right) = \frac{d(\delta\mathbf{u})}{dx}$$



# PMPE As a Variation

- Necessary condition for minimum PE
  - Stationary condition <--> first variation = 0

$$\delta\Pi(\mathbf{u};\bar{\mathbf{u}}) = \lim_{\tau \to 0}\frac{1}{\tau}[\Pi(\mathbf{u} + \tau\bar{\mathbf{u}}) - \Pi(\mathbf{u})] = \frac{d}{d\tau}\Pi(\mathbf{u} + \tau\bar{\mathbf{u}})\Big|_{\tau=0} = 0$$

for all $\bar{\mathbf{u}} \in \mathbb{Z}$

- Variation of strain energy

$$\delta\left(\frac{\partial\mathbf{u}}{\partial\mathbf{x}}\right) = \frac{d}{d\tau}\left(\frac{\partial\mathbf{u} + \tau\bar{\mathbf{u}}}{\partial\mathbf{x}}\right)\Big|_{\tau=0} = \frac{\partial\bar{\mathbf{u}}}{\partial\mathbf{x}}$$

$$\delta\varepsilon(\mathbf{u}) = \varepsilon(\bar{\mathbf{u}}) = \bar{\varepsilon} \qquad \delta\sigma = \mathbf{D} : \bar{\varepsilon}$$

$$\begin{aligned}\delta U(\mathbf{u};\bar{\mathbf{u}}) &= \tfrac{1}{2}\iint_\Omega \left[\varepsilon(\bar{\mathbf{u}}) : \mathbf{D} : \varepsilon(\mathbf{u}) + \varepsilon(\mathbf{u}) : \mathbf{D} : \varepsilon(\bar{\mathbf{u}})\right] d\Omega \\ &= \iint_\Omega \varepsilon(\bar{\mathbf{u}}) : \mathbf{D} : \varepsilon(\mathbf{u})\, d\Omega \\ &\equiv a(\mathbf{u},\bar{\mathbf{u}}) \qquad \text{Energy bilinear form}\end{aligned}$$

## PMPE As a Variation cont.

- Variation of work done by applied loads

$$\delta W(\mathbf{u};\bar{\mathbf{u}}) = \iint_{\Omega} \bar{\mathbf{u}} \cdot \mathbf{f}^b \, d\Omega + \int_{\Gamma^s} \bar{\mathbf{u}} \cdot \mathbf{t} \, d\Gamma \equiv \ell(\bar{\mathbf{u}})$$ Load linear form

$$\delta\Pi(\mathbf{u};\bar{\mathbf{u}}) = \delta U(\mathbf{u};\bar{\mathbf{u}}) - \delta W(\mathbf{u};\bar{\mathbf{u}}) = 0$$

- Thus, PMPE becomes

$$\boxed{a(\mathbf{u},\bar{\mathbf{u}}) = \ell(\bar{\mathbf{u}}), \qquad \forall \bar{\mathbf{u}} \in \mathbb{Z}}$$

  - Load form $\ell(\bar{\mathbf{u}})$ is linear with respect to $\bar{\mathbf{u}}$
  - Energy form $a(\mathbf{u}, \bar{\mathbf{u}})$ is symmetric, bilinear w.r.t. $\mathbf{u}$ and $\bar{\mathbf{u}}$
  - Different problems have different $a(\mathbf{u}, \bar{\mathbf{u}})$ and $\ell(\bar{\mathbf{u}})$, but they share the same property
- **How can we satisfy "for all $\bar{\mathbf{u}} \in \mathbb{Z}$" requirement? Can we test an infinite number of $\bar{\mathbf{u}}$?**



## Example – Uniaxial Bar

- Assumed displacement u(x) = cx →
  - virtual displacement is in the same space with u(x): $\bar{u}(x) = \bar{c}x$
- Variation of strain energy

$$\delta U = \frac{d}{d\tau}\left[\frac{1}{2}\int_0^L EA\left[(u+\tau\bar{u})'\right]^2 dx\right]_{\tau=0} = \frac{1}{2}\int_0^L 2EA(u+\tau\bar{u})'\bar{u}'\,dx\Big|_{\tau=0}$$
$$= \int_0^L EAu'\bar{u}'\,dx = EALc\bar{c}$$

- Variation of applied load

$$\delta W = \frac{d}{d\tau}\left[F\left[u(L)+\tau\bar{u}(L)\right]\right]_{\tau=0} = F\bar{u}(L) = FL\bar{c}$$

- PMPE

$$\delta\Pi = \delta U - \delta W = \bar{c}(EALc - FL) = 0 \qquad u(x) = cx = \frac{Fx}{EA}$$

# Principle of Virtual Work

- Instead of solving the strong form directly, we want to solve the equation with relaxed requirement (**weak form**)
- **Virtual work** – Work resulting from real forces acting through a virtual displacement
- **Principle of virtual work** – when a system is in equilibrium, the forces applied to the system will not produce any virtual work for **arbitrary** virtual displacements
  - Balance of linear momentum is force equilibrium $\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}^b = 0$
  - Thus, the virtual work can be obtained by multiplying the force equilibrium equation with a virtual displacement

$$\bar{W} = \iint_\Omega (\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}^b) \cdot \bar{\mathbf{u}} \, d\Omega$$

  - If the above virtual work becomes zero for arbitrary $\bar{\mathbf{u}}$, then it satisfies the original equilibrium equation in a **weak sense**



# Principle of Virtual Work cont

- PVW $\iint_\Omega (\sigma_{ij,i} + f_j^b)\bar{u}_j \, d\Omega = 0 \quad \forall \bar{\mathbf{u}} \in \mathbb{Z}$

$$= \iint_\Omega \sigma_{ij,i}\bar{u}_j \, d\Omega + \iint_\Omega f_j^b \bar{u}_j \, d\Omega$$

  - Integration-by-parts

$$= \iint_\Omega \left[ (\sigma_{ij}\bar{u}_j)_{,i} - \sigma_{ij}\bar{u}_{j,i} \right] d\Omega + \iint_\Omega f_j^b \bar{u}_j \, d\Omega$$

  - Divergence Thm

$$= \int_\Gamma n_i \sigma_{ij} \bar{u}_j \, d\Gamma - \iint_\Omega \sigma_{ij}\bar{u}_{j,i} \, d\Omega + \iint_\Omega f_j^b \bar{u}_j \, d\Omega$$

  - The boundary is decomposed by $\Gamma = \Gamma^h \cup \Gamma^s$

$$\bar{u}_j = 0 \text{ on } \Gamma^h \text{ and } n_i \sigma_{ij} = t_j \text{ on } \Gamma^s$$

$$= \int_{\Gamma^s} t_j \bar{u}_j \, d\Gamma - \iint_\Omega \sigma_{ij}\bar{u}_{j,i} \, d\Omega + \iint_\Omega f_j^b \bar{u}_j \, d\Omega$$

# Principle of Virtual Work cont

- Since $\sigma_{ij}$ is symmetric

$$\sigma_{ij}\bar{u}_{j,i} = \sigma_{ij}\mathrm{sym}(\bar{u}_{j,i}) = \sigma_{ij}\bar{\varepsilon}_{ij} \qquad \mathrm{sym}(\bar{u}_{i,j}) = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial X_j} + \frac{\partial \bar{u}_j}{\partial X_i}\right) = \bar{\varepsilon}_{ij}$$

- **Weak Form** of BVP

$$\boxed{\iint_\Omega \sigma_{ij}\bar{\varepsilon}_{ij}\, d\Omega = \iint_\Omega f_j^b \bar{u}_j\, d\Omega + \int_{\Gamma^s} t_j \bar{u}_j\, d\Gamma \quad \forall \bar{\mathbf{u}} \in \mathbb{Z}}$$

**Internal virtual work = external virtual work**

Starting point of FEM

- Symbolic expression

$$\boxed{a(\mathbf{u}, \bar{\mathbf{u}}) = \ell(\bar{\mathbf{u}}) \quad \forall \bar{\mathbf{u}} \in \mathbb{Z}} \Longrightarrow [\mathbf{K}]\{\mathbf{d}\} = \{\mathbf{F}\}$$

FE equation

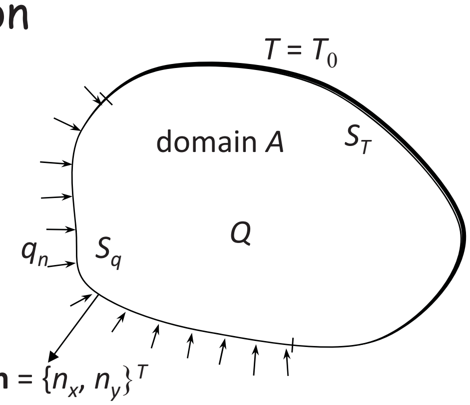
- **Energy form**: $a(\mathbf{u}, \bar{\mathbf{u}}) = \iint_\Omega \sigma : \bar{\varepsilon}\, d\Omega$
- **Load form**: $\ell(\bar{\mathbf{u}}) = \iint_\Omega \rho \bar{\mathbf{u}} \cdot \mathbf{f}^b\, d\Omega + \int_{\Gamma^s} \bar{\mathbf{u}} \cdot \mathbf{t}\, d\Gamma$



# Example – Heat Transfer Problem

- Steady-State Differential Equation

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) + Q = 0$$




- Boundary conditions

$$\begin{cases} T = T_0 \text{ on } S_T \\ q_n = -n_x k_x \dfrac{dT}{dx} - n_y k_y \dfrac{dT}{dy} \quad \text{on } S_q \end{cases}$$

- Space of kinematically admissible temperature

$$\mathbb{Z} = \left\{ \bar{T} \in H^1(\Omega) \,\middle|\, \bar{T}(\mathbf{x}) = 0, \forall \mathbf{x} \in S_T \right\}$$

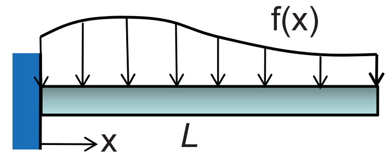
- Multiply by virtual temperature, integrate by part, and apply boundary conditions

$$\int_\Omega \left( k_x \frac{\partial T}{\partial x}\frac{\partial \bar{T}}{\partial x} + k_y \frac{\partial T}{\partial y}\frac{\partial \bar{T}}{\partial y} \right) d\Omega = \int_\Omega \bar{T} Q\, d\Omega + \int_{S_q} \bar{T} q_n\, dS_q, \quad \forall \bar{T} \in \mathbb{Z}$$

# Example – Beam Problem

- Governing DE

$$EI\frac{d^4 v}{dx^4} = f(x), \quad x \in [0,L]$$




- Boundary conditions for cantilevered beam

$$v(0) = \frac{dv}{dx}(0) = \frac{d^2 v}{dx^2}(L) = \frac{d^3 v}{dx^3}(L) = 0$$

- Space of kinematically admissible displacement

$$\mathbb{Z} = \left\{ \bar{v} \in H^2[0,L] \,\middle|\, \bar{v}(0) = \frac{d\bar{v}}{dx}(0) = 0 \right\}$$

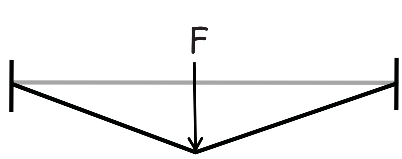
- Integrate-by-part twice, and apply BCs

$$\int_0^L EI\frac{d^2\bar{v}}{dx^2}\frac{d^2 v}{dx^2}dx = \int_0^L f\bar{v}\,dx, \quad \forall \bar{v} \in \mathbb{Z}$$



# Difference b/w Strong and Weak Solutions

- The solution of the strong form needs to be **twice differentiable** $\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right) + Q = 0$
- The solution of the weak form requires the **first-order derivatives are integrable** $\Rightarrow$ bigger solution space than that of the strong form $\int_\Omega \left(k_x\frac{\partial T}{\partial x}\frac{\partial \bar{T}}{\partial x} + k_y\frac{\partial T}{\partial y}\frac{\partial \bar{T}}{\partial y}\right)d\Omega$
- If the strong form has a solution, it is the solution of the weak form
- If the strong form does not have a solution, the weak form may have a natural solution



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# FINITE ELEMENT METHOD



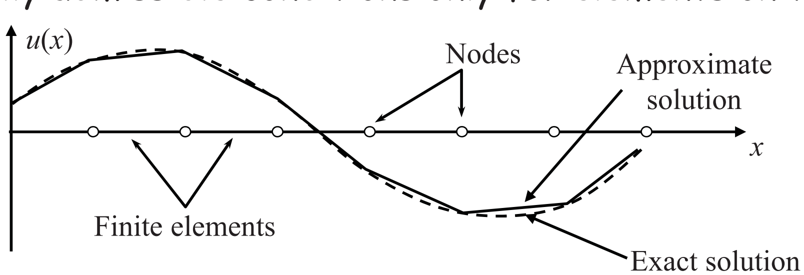
## Finite Element Approximation

- Difficult to solve a variational equation analytically
- Approximate solution
  - Linear combination of trial functions
  - Smoothness & accuracy depend on the choice of trial functions
  - If the approximate solution is expressed in the entire domain, it is difficult to satisfy kinematically admissible conditions

$$u(x) = \sum_{i=1}^{n} c_i \phi_i(x)$$

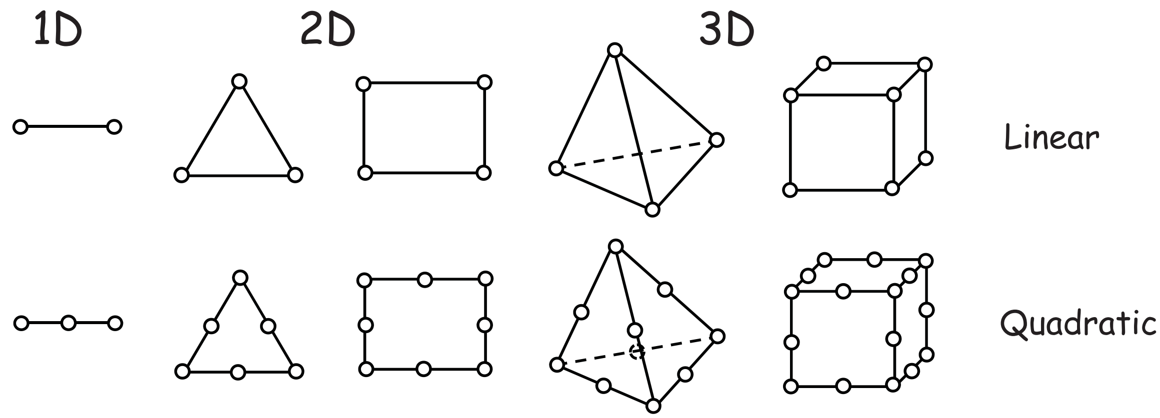
- Finite element approximation
  - Approximate solution in simple sub-domains (elements)
  - Simple trial functions (low-order polynomials) within an element
  - Kinematically admissible conditions only for elements on the boundary

Piecewise-linear approximation



# Finite Elements

- Types of finite elements




- Variational equation is imposed on each element.

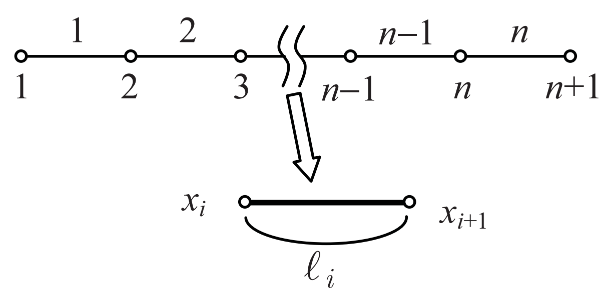
$$\int_0^1 \square\, dx = \int_0^{0.1} \square\, dx + \int_{0.1}^{0.2} \square\, dx + \cdots + \int_{0.9}^{1} \square\, dx$$

One element



# Trial Solution

- Solution within an element is approximated using simple polynomials.




- *i*-th element is composed of two nodes: $x_i$ and $x_{i+1}$. Since two unknowns are involved, linear polynomial can be used:

$$u(x) = a_0 + a_1 x, \quad x_i \le x \le x_{i+1}$$

- The unknown coefficients, $a_0$ and $a_1$, will be expressed in terms of nodal solutions $u(x_i)$ and $u(x_{i+1})$.

# Trial Solution *cont.*

- Substitute two nodal values

$$\begin{cases} u(x_i) = u_i = a_0 + a_1 x_i \\ u(x_{i+1}) = u_{i+1} = a_0 + a_1 x_{i+1} \end{cases}$$

- Express $a_0$ and $a_1$ in terms of $u_i$ and $u_{i+1}$. Then, the solution is approximated by

$$u(x) = \underbrace{\frac{x_{i+1} - x}{L^{(e)}}}_{N_1(x)} u_i + \underbrace{\frac{x - x_i}{L^{(e)}}}_{N_2(x)} u_{i+1}$$

- Solution for Element e:

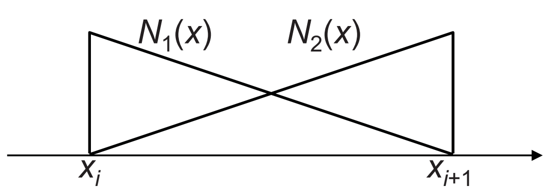
$$u(x) = N_1(x) u_i + N_2(x) u_{i+1}, \quad x_i \le x \le x_{i+1}$$

- $N_1(x)$ and $N_2(x)$: **Shape Function** or **Interpolation Function**

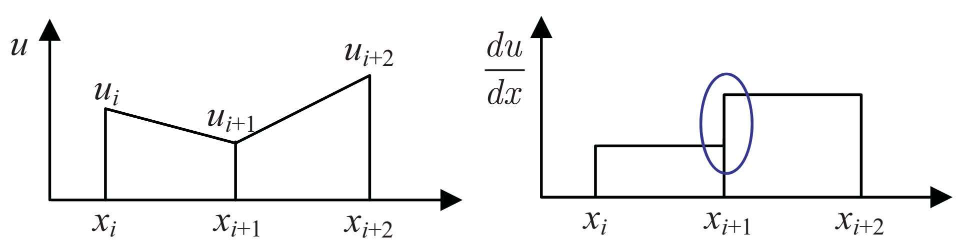


# Trial Solution *cont.*

- Observations
  - Solution u(x) is interpolated using its nodal values $u_i$ and $u_{i+1}$.
  - $N_1(x)$ = 1 at node $x_i$, and =0 at node $x_{i+1}$.




  - The solution is approximated by piecewise linear polynomial and its gradient is constant within an element.




  - Stress and strain (derivative) are often averaged at the node.

# 1D Finite Elements

- 1D BVP $\dfrac{d^2u}{dx^2} + p(x) = 0, \quad 0 \le x \le 1$

$$\left.\begin{aligned} &u(0) = 0 \\ &\frac{du}{dx}(1) = 0 \end{aligned}\right\} \quad \text{Boundary conditions}$$

- Use PVW $\int_0^1 \left( \dfrac{d^2u}{dx^2} + p \right) \bar{u}\, dx = 0$

Space of kinematically admissible displacements

$$\mathbb{Z} = \left\{ \bar{u} \in H^{(1)}[0,1] \,\middle|\, \bar{u}(0) = 0 \right\}$$

- Integration-by-parts

$$\left. \frac{du}{dx}\bar{u} \right|_0^1 - \int_0^1 \frac{du}{dx}\frac{d\bar{u}}{dx}\, dx = -\int_0^1 p\bar{u}\, dx$$

  - This variational equation also satisfies at individual element level

(1) $$\boxed{\int_{x_i}^{x_j} \frac{du}{dx}\frac{d\bar{u}}{dx}\, dx = \int_{x_i}^{x_j} p\bar{u}\, dx} \qquad \forall \bar{\mathbf{u}} \in \mathbb{Z}$$



# 1D Interpolation Functions

- Finite element approximation for one element (e) at a time

$$u^{(e)}(x) = u_i N_1(x) + u_{i+1} N_2(x) = \mathbf{N}^{(e)} \cdot \mathbf{d}^{(e)}$$

$$\mathbf{d}^{(e)} = \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} \qquad \mathbf{N}^{(e)} = \lfloor N_1 \quad N_2 \rfloor$$

- Satisfies interpolation condition

$$u^{(e)}(x_i) = u_i$$
$$u^{(e)}(x_{i+1}) = u_{i+1}$$

- Interpolation of displacement variation (same with u)

$$\bar{u}^{(e)}(x) = \bar{u}_i N_1(x) + \bar{u}_{i+1} N_2(x) = \mathbf{N}^{(e)} \cdot \bar{\mathbf{d}}^{(e)}$$

- Derivative of u(x): differentiating interpolation functions

$$\frac{du^{(e)}}{dx} = \left\lfloor \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right\rfloor \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} = \left\lfloor -\frac{1}{L^{(e)}} \quad \frac{1}{L^{(e)}} \right\rfloor \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} = \mathbf{B}^{(e)} \cdot \mathbf{d}^{(e)}$$

# Element-Level Variational Equation

- Approximate variational equation (1) for element (e)

$$\bar{\mathbf{d}}^{(e)T}\left[\int_{x_i}^{x_j}\mathbf{B}^{(e)T}\mathbf{B}^{(e)}\,dx\right]\mathbf{d}^{(e)} = \bar{\mathbf{d}}^{(e)T}\int_{x_i}^{x_j}\mathbf{N}^{(e)T}p(x)\,dx + \bar{\mathbf{d}}^{(e)T}\begin{Bmatrix} -\dfrac{du}{dx}(x_i) \\ +\dfrac{du}{dx}(x_{i+1}) \end{Bmatrix}$$

  - Must satisfied for all $\bar{u}^{(e)}(x) \in \mathbb{Z}$
  - If element (e) is not on the boundary, $\bar{\mathbf{d}}^{(e)}$ can be arbitrary

- Element-level variational equation

$$\left[\boxed{\int_{x_i}^{x_j}\mathbf{B}^{(e)T}\mathbf{B}^{(e)}\,dx}\right]\mathbf{d}^{(e)} = \boxed{\int_{x_i}^{x_j}\mathbf{N}^{(e)T}p(x)\,dx} + \begin{Bmatrix} -\dfrac{du}{dx}(x_i) \\ +\dfrac{du}{dx}(x_{i+1}) \end{Bmatrix}$$

2x2 matrix

2x1 vector

$$[\mathbf{k}^{(e)}]\{\mathbf{d}^{(e)}\} = \left\{\mathbf{f}^{(e)}\right\} + \begin{Bmatrix} -\dfrac{du}{dx}(x_i) \\ +\dfrac{du}{dx}(x_{i+1}) \end{Bmatrix}$$



# Assembly

- Need to derive the element-level equation for all elements
- Consider Elements 1 and 2 (connected at Node 2)

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{(1)} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}^{(1)} + \begin{Bmatrix} -\dfrac{du}{dx}(x_1) \\ +\dfrac{du}{dx}(x_2) \end{Bmatrix}$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{(2)} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_2 \\ f_3 \end{Bmatrix}^{(2)} + \begin{Bmatrix} -\dfrac{du}{dx}(x_2) \\ +\dfrac{du}{dx}(x_3) \end{Bmatrix}$$

- Assembly

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} + \begin{Bmatrix} -\dfrac{du}{dx}(x_1) \\ 0 \\ \dfrac{du}{dx}(x_3) \end{Bmatrix}$$

Vanished unknown term

# Assembly cont.

- Assembly of $N_E$ elements ($N_D = N_E + 1$)

$$\underbrace{\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & \cdots & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & \cdots & 0 \\ 0 & k_{221}^{(2)} & k_{22}^{(2)} + k_{11}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & k_{21}^{(N_E)} & k_{22}^{(N_E)} \end{bmatrix}}_{(N_D \times N_D)} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{Bmatrix}}_{(N_D \times 1)} = \underbrace{\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} + f_3^{(3)} \\ \vdots \\ f_N^{(N_E)} \end{Bmatrix}}_{(N_D \times 1)} + \underbrace{\begin{Bmatrix} -\dfrac{du}{dx}(x_1) \\ 0 \\ 0 \\ \vdots \\ +\dfrac{du}{dx}(x_N) \end{Bmatrix}}_{(N_D \times 1)}$$

$$\boxed{[\mathbf{K}]\{\mathbf{d}\} = \{\mathbf{F}\}}$$

- Coefficient matrix [K] is singular; it will become non-singular after applying boundary conditions



# Example

- Use three equal-length elements

$$\frac{d^2u}{dx^2} + x = 0, \ \ 0 \le x \le 1 \qquad u(0) = 0, \ \ u(1) = 0$$

- All elements have the same coefficient matrix

$$\left[\mathbf{k}^{(e)}\right]_{2\times 2} = \frac{1}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}, \quad (e = 1,2,3)$$

- RHS ($p(x) = x$)

$$\{\mathbf{f}^{(e)}\} = \int_{x_i}^{x_{i+1}} p(x) \begin{Bmatrix} N_1(x) \\ N_2(x) \end{Bmatrix} dx = \frac{1}{L^{(e)}} \int_{x_i}^{x_{i+1}} \begin{Bmatrix} x(x_{i+1} - x) \\ x(x - x_i) \end{Bmatrix} dx$$

$$= L^{(e)} \begin{Bmatrix} \dfrac{x_i}{3} + \dfrac{x_{i+1}}{6} \\ \dfrac{x_i}{6} + \dfrac{x_{i+1}}{3} \end{Bmatrix}, \quad (e = 1,2,3)$$

# Example cont.

- RHS cont. $\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = \frac{1}{54}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$, $\begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \frac{1}{54}\begin{Bmatrix} 4 \\ 5 \end{Bmatrix}$, $\begin{Bmatrix} f_3^{(3)} \\ f_4^{(3)} \end{Bmatrix} = \frac{1}{54}\begin{Bmatrix} 7 \\ 8 \end{Bmatrix}$
- Assembly

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 3+3 & -3 & 0 \\ 0 & -3 & 3+3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{54} - \frac{du}{dx}(0) \\ \frac{2}{54} + \frac{4}{54} \\ \frac{7}{54} + \frac{5}{54} \\ \frac{8}{54} + \frac{du}{dx}(1) \end{Bmatrix}$$

Element 1
Element 2
Element 3

- Apply boundary conditions
  - Deleting 1st and 4th rows and columns

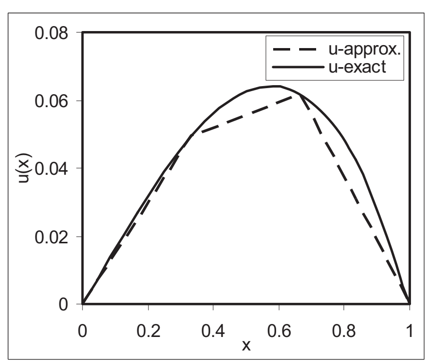
$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{9}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \Longrightarrow \begin{aligned} u_2 &= \frac{4}{81} \\ u_3 &= \frac{5}{81} \end{aligned}$$



# EXAMPLE cont.

- Approximate solution

$$u(x) = \begin{cases} \frac{4}{27}x, & 0 \le x \le \frac{1}{3} \\ \frac{4}{81} + \frac{1}{27}\left(x - \frac{1}{3}\right), & \frac{1}{3} \le x \le \frac{2}{3} \\ \frac{5}{81} - \frac{5}{27}\left(x - \frac{2}{3}\right), & \frac{2}{3} \le x \le 1 \end{cases}$$




- Exact solution

$$u(x) = \frac{1}{6}x\left(1 - x^2\right)$$

  - Three element solutions are poor
  - Need more elements

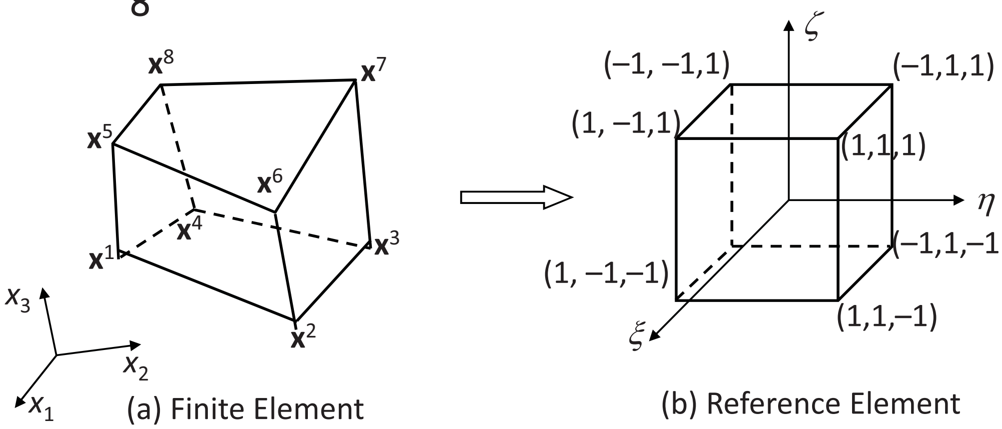
# 3D Solid Element

- Isoparametric mapping
  - Build interpolation functions on the reference element
  - Jacobian: mapping relation between physical and reference elem.
- Interpolation and mapping

$$\mathbf{u}(\xi) = \sum_{I=1}^{8} N_I(\xi)\mathbf{u}_I \qquad \mathbf{x}(\xi) = \sum_{I=1}^{8} N_I(\xi)\mathbf{x}_I$$

Same for mapping and interpolation

$$N_I(\xi) = \frac{1}{8}(1+\xi\xi_I)(1+\eta\eta_I)(1+\zeta\zeta_I)$$




(a) Finite Element

(b) Reference Element



# 3D Solid Element cont.

- Jacobian matrix

$$\mathbf{J}_{3\times3} = \frac{\partial \mathbf{x}}{\partial \xi} = \sum_{I=1}^{8} \mathbf{x}_I \frac{\partial N_I(\xi)}{\partial \xi}$$

$|\mathbf{J}|$ : Jacobian

- Derivatives of shape functions

$$\begin{Bmatrix} \frac{\partial N_I}{\partial \xi} & \frac{\partial N_I}{\partial \eta} & \frac{\partial N_I}{\partial \zeta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial N_I}{\partial x_1} & \frac{\partial N_I}{\partial x_2} & \frac{\partial N_I}{\partial x_3} \end{Bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial \xi} & \frac{\partial x_1}{\partial \eta} & \frac{\partial x_1}{\partial \zeta} \\ \frac{\partial x_2}{\partial \xi} & \frac{\partial x_2}{\partial \eta} & \frac{\partial x_2}{\partial \zeta} \\ \frac{\partial x_3}{\partial \xi} & \frac{\partial x_3}{\partial \eta} & \frac{\partial x_3}{\partial \zeta} \end{bmatrix}$$

$$\frac{\partial N_I}{\partial \xi} = \frac{\partial N_I}{\partial \mathbf{x}} \cdot \mathbf{J}$$

$$\frac{\partial N_I}{\partial \mathbf{x}} = \frac{\partial N_I}{\partial \xi} \cdot \mathbf{J}^{-1}$$

  - Jacobian should not be zero anywhere in the element
  - Zero or negative Jacobian: mapping is invalid (bad element shape)

# 3D Solid Element cont.

- Displacement-strain relation

$$\varepsilon(\mathbf{u}) = \sum_{I=1}^{8} \mathbf{B}_I \mathbf{u}_I$$

$$\bar{\varepsilon} = \varepsilon(\bar{\mathbf{u}}) = \sum_{I=1}^{8} \mathbf{B}_I \bar{\mathbf{u}}_I$$

$$\mathbf{B}_I = \begin{bmatrix} N_{I,1} & 0 & 0 \\ 0 & N_{I,2} & 0 \\ 0 & 0 & N_{I,3} \\ N_{I,2} & N_{I,1} & 0 \\ 0 & N_{I,3} & N_{I,2} \\ N_{I,3} & 0 & N_{I,1} \end{bmatrix}$$

$$N_{I,i} = \frac{\partial N_I}{\partial x_i}$$



# 3D Solid Element cont.

- Transformation of integration domain

$$\iiint_{\Omega} d\Omega = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |\mathbf{J}|\, d\xi d\eta d\zeta$$

- Energy form

$$a(\mathbf{u},\bar{\mathbf{u}}) = \sum_{I=1}^{8}\sum_{J=1}^{8} \bar{\mathbf{u}}_I^T \left[ \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}_I^T \mathbf{D} \mathbf{B}_J |\mathbf{J}|\, d\xi d\eta d\zeta \right] \mathbf{u}_J \equiv \{\bar{\mathbf{d}}\}^T [\mathbf{k}] \{\mathbf{d}\}$$

- Load form

$$\ell(\bar{\mathbf{u}}) = \sum_{I=1}^{8} \bar{\mathbf{u}}_I^T \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} N_I(\xi) \mathbf{f}^b |\mathbf{J}|\, d\xi d\eta d\zeta = \{\bar{\mathbf{d}}\}^T \{\mathbf{f}\}$$

- Discrete variational equation

$$\{\bar{\mathbf{d}}\}^T [\mathbf{k}] \{\mathbf{d}\} = \{\bar{\mathbf{d}}\}^T \{\mathbf{f}\}, \quad \forall \{\bar{\mathbf{d}}\} \in \mathbb{Z}_h$$

# Numerical Integration

- For bar and beam, analytical integration is possible
- For plate and solid, analytical integration is difficult, if not impossible
- Gauss quadrature is most popular in FEM due to simplicity and accuracy
- 1D Gauss quadrature

$$\int_{-1}^{1} f(\xi)\,d\xi \approx \sum_{i=1}^{NG} \omega_i f(\xi_i)$$

  - NG: No. of integ. points; $\xi_i$: integ. point; $\omega_i$: integ. weight
  - **$\xi_i$ and $\omega_i$ are chosen so that the integration is exact for (2*NG – 1)-order polynomial**
  - Works well for smooth function
  - Integration domain is [-1, 1]



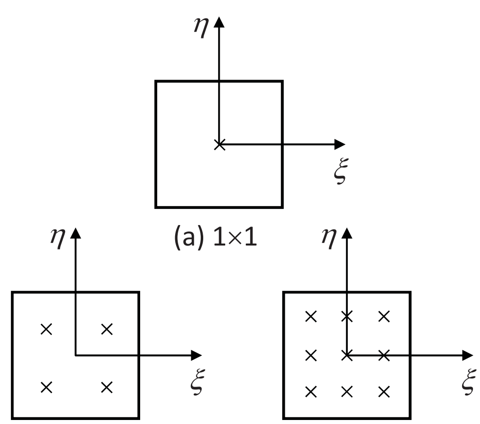
# Numerical Integration cont.

- Multi-dimensions

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi d\eta = \sum_{i=1}^{NG}\sum_{j=1}^{NG} \omega_i\omega_j f(\xi_i,\eta_j)$$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta)\,d\xi d\eta d\zeta = \sum_{i=1}^{NG}\sum_{j=1}^{NG}\sum_{k=1}^{NG} \omega_i\omega_j\omega_k f(\xi_i,\eta_j,\zeta_k)$$

| *NG* | Integration Points ($\xi_i$) | Weights ($\omega_i$) |
|---|---|---|
| 1 | 0.0 | 2.0 |
| 2 | ±.5773502692 | 1.0 |
| 3 | ±.7745966692 | .5555555556 |
| | 0.0 | .8888888889 |
| 4 | ±.8611363116 | .3478546451 |
| | ±.3399810436 | .6521451549 |
| 5 | ±.9061798459 | .2369268851 |
| | ±.5384693101 | .4786286705 |
| | 0.0 | .5688888889 |




(a) 1×1 (b) 2×2 (c) 3×3

# ELAST3D.m

- A module to solve linear elastic problem using NLFEA.m
- Input variables for ELAST3D.m

| Variable | Array size | Meaning |
|---|---|---|
| ETAN | (6,6) | Elastic stiffness matrix Eq. (1.81) |
| UPDATE | Logical variable | If true, save stress values |
| LTAN | Logical variable | If true, calculate the global stiffness matrix |
| NE | Integer | Total number of elements |
| NDOF | Integer | Dimension of problem (3) |
| XYZ | (3,NNODE) | Coordinates of all nodes |
| LE | (8,NE) | Element connectivity |



# How to Solve Linear Problem in Nonlinear Code

- Linear matrix solver

$$[\mathbf{K}]\{\mathbf{d}\} = \{\mathbf{F}\} \Longrightarrow \{\mathbf{f}^{\text{int}}\} = \{\mathbf{f}^{\text{ext}}\} \Longrightarrow \{\mathbf{f}\} = \{\mathbf{f}^{\text{ext}}\} - \{\mathbf{f}^{\text{int}}\} = \{\mathbf{0}\}$$

  - Construct stiffness matrix and force vector
  - Use LU decomposition to solve for unknown displacement $\{\mathbf{d}\}$
- Nonlinear solver (iterative solver)
  - Assume the solution at iteration n is known, and n+1 is unknown

$$\{\mathbf{d}^{n+1}\} = \{\mathbf{d}^{n}\} + \{\Delta\mathbf{d}\}$$

For linear problem, $\{\mathbf{d}^n\} = \{\mathbf{0}\}$

$$\{\mathbf{f}^{n+1}\} = \{\mathbf{f}^{n}\} + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{d}}\right]\{\Delta\mathbf{d}\} \approx \{\mathbf{0}\}$$

$$\Longrightarrow \{\mathbf{F}\} - [\mathbf{K}]\cancel{\{\mathbf{d}^{n}\}} - [\mathbf{K}]\{\Delta\mathbf{d}\} = 0$$

$$\Longrightarrow [\mathbf{K}]\{\Delta\mathbf{d}\} = \{\mathbf{F}\}$$

Only one iteration!!

```
function ELAST3D(ETAN, UPDATE, LTAN, NE, NDOF, XYZ, LE)
%***********************************************************************
%  MAIN PROGRAM COMPUTING GLOBAL STIFFNESS MATRIX AND RESIDUAL FORCE FOR
%  ELASTIC MATERIAL MODELS
%***********************************************************************
%%
  global DISPTD FORCE GKF SIGMA
  %
  % Integration points and weights (2-point integration)
  XG=[-0.57735026918963D0, 0.57735026918963D0];
  WGT=[1.00000000000000D0, 1.00000000000000D0];
  %
  % Index for history variables (each integration pt)
  INTN=0;
  %
  %LOOP OVER ELEMENTS, THIS IS MAIN LOOP TO COMPUTE K AND F
  for IE=1:NE
    % Nodal coordinates and incremental displacements
    ELXY=XYZ(LE(IE,:),:);
    % Local to global mapping
    IDOF=zeros(1,24);
    for I=1:8
      II=(I-1)*NDOF+1;
      IDOF(II:II+2)=(LE(IE,I)-1)*NDOF+1:(LE(IE,I)-1)*NDOF+3;
    end
    DSP=DISPTD(IDOF);
    DSP=reshape(DSP,NDOF,8);
    %
    %LOOP OVER INTEGRATION POINTS
    for LX=1:2, for LY=1:2, for LZ=1:2
      E1=XG(LX); E2=XG(LY); E3=XG(LZ);
      INTN = INTN + 1;
      %
      % Determinant and shape function derivatives
      [~, SHPD, DET] = SHAPEL([E1 E2 E3], ELXY);
      FAC=WGT(LX)*WGT(LY)*WGT(LZ)*DET;
```



```
      % Strain
      DEPS=DSP*SHPD';
      DDEPS=[DEPS(1,1) DEPS(2,2) DEPS(3,3) ...
             DEPS(1,2)+DEPS(2,1) DEPS(2,3)+DEPS(3,2) DEPS(1,3)+DEPS(3,1)]';
      %
      % Stress
      STRESS = ETAN*DDEPS;
      %
      % Update stress
      if UPDATE
        SIGMA(:,INTN)=STRESS;
        continue;
      end
      %
      % Add residual force and stiffness matrix
      BM=zeros(6,24);
      for I=1:8
        COL=(I-1)*3+1:(I-1)*3+3;
        BM(:,COL)=[SHPD(1,I) 0         0;
                   0         SHPD(2,I) 0;
                   0         0         SHPD(3,I);
                   SHPD(2,I) SHPD(1,I) 0;
                   0         SHPD(3,I) SHPD(2,I);
                   SHPD(3,I) 0         SHPD(1,I)];
      end
      %
      % Residual forces
      FORCE(IDOF) = FORCE(IDOF) - FAC*BM'*STRESS;
      %
      % Tangent stiffness
      if LTAN
        EKF = BM'*ETAN*BM;
        GKF(IDOF,IDOF)=GKF(IDOF,IDOF)+FAC*EKF;
      end
    end, end, end, end
end
```

```
function [SF, GDSF, DET] = SHAPEL(XI, ELXY)
%***********************************************************************
% Compute shape function, derivatives, and determinant of hexahedral
element
%***********************************************************************
%%
  XNODE=[-1  1  1 -1 -1  1  1 -1;
         -1 -1  1  1 -1 -1  1  1;
         -1 -1 -1 -1  1  1  1  1];
  QUAR = 0.125;
  SF=zeros(8,1);
  DSF=zeros(3,8);
  for I=1:8
    XP = XNODE(1,I);
    YP = XNODE(2,I);
    ZP = XNODE(3,I);
    %
    XI0 = [1+XI(1)*XP 1+XI(2)*YP 1+XI(3)*ZP];
    %
    SF(I)  = QUAR*XI0(1)*XI0(2)*XI0(3);
    DSF(1,I)  = QUAR*XP*XI0(2)*XI0(3);
    DSF(2,I)  = QUAR*YP*XI0(1)*XI0(3);
    DSF(3,I)  = QUAR*ZP*XI0(1)*XI0(2);
  end
  GJ = DSF*ELXY;
  DET = det(GJ);
  GJINV=inv(GJ);
  GDSF=GJINV*DSF;
end
```

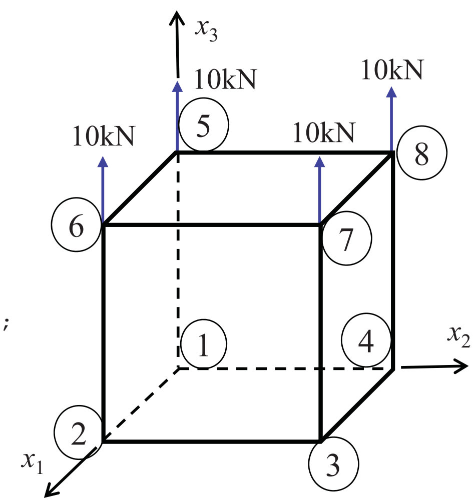
SF($8\times1$): shape functions,

GDSF ($3\times8$): shape functions derivatives

DET: Jacobian of the mapping



# One Element Tension Example




```
%
% One element example
%
% Nodal coordinates
XYZ=[0 0 0;1 0 0;1 1 0;0 1 0;0 0 1;1 0 1;1 1 1;0 1 1];
%
% Element connectivity
LE=[1 2 3 4 5 6 7 8];
%
% External forces [Node, DOF, Value]
EXTFORCE=[5 3 10.0E3; 6 3 10.0E3; 7 3 10.0E3; 8 3 10.0E3];
%
% Prescribed displacements [Node, DOF, Value]
SDISPT=[1 1 0;1 2 0;1 3 0;2 2 0;2 3 0;3 3 0;4 1 0;4 3 0];
%
% Material properties
% MID:0(Linear elastic) PROP=[LAMBDA NU]
MID=0;
PROP=[110.747E3 80.1938E3];
%
% Load increments [Start End Increment InitialFactor FinalFactor]
TIMS=[0.0 1.0 1.0 0.0 1.0]';
%
% Set program parameters
ITRA=30; ATOL=1.0E5; NTOL=6; TOL=1E-6;
%
% Calling main function
NOUT = fopen('output.txt','w');
NLFEA(ITRA, TOL, ATOL, NTOL, TIMS, NOUT, MID, PROP, EXTFORCE, SDISPT, XYZ, LE);
fclose(NOUT);
```

# One Element Output

## Command line output

```
  Time     Time step   Iter    Residual
 1.00000  1.000e+00     2    5.45697e-12
```

## Contents in output.txt

```
TIME =   1.000e+00

Nodal Displacements

 Node          U1          U2          U3
    1   0.000e+00   0.000e+00   0.000e+00
    2  -5.607e-08   0.000e+00   0.000e+00
    3  -5.607e-08  -5.607e-08   0.000e+00
    4   0.000e+00  -5.607e-08   0.000e+00
    5  -5.494e-23   1.830e-23   1.933e-07
    6  -5.607e-08   4.061e-23   1.933e-07
    7  -5.607e-08  -5.607e-08   1.933e-07
    8  -8.032e-23  -5.607e-08   1.933e-07

Element Stress

        S11         S22         S33         S12         S23         S13
Element     1
  0.000e+00   1.091e-11   4.000e+04  -2.322e-13   6.633e-13  -3.317e-12
  0.000e+00   0.000e+00   4.000e+04  -3.980e-13   1.327e-13  -9.287e-13
 -3.638e-12   7.276e-12   4.000e+04  -1.592e-12  -2.123e-12  -3.317e-12
  0.000e+00   0.000e+00   4.000e+04   2.653e-13  -2.123e-12   5.307e-13
  0.000e+00   0.000e+00   4.000e+04   5.638e-13   3.449e-12  -1.327e-12
  0.000e+00   0.000e+00   4.000e+04  -1.194e-12   4.776e-12   1.061e-12
  0.000e+00   0.000e+00   4.000e+04  -7.960e-13   2.919e-12  -3.449e-12
  3.638e-12   3.638e-12   4.000e+04  -5.307e-13   3.715e-12   1.061e-12
                    *** Successful end of program ***
```